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6. AUTHOR(S) Chris Rollins, David Resendes, and Michael R. Squillante

ASL POC: Harry Auvermann

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This report summarizes the work performed under DOD SBIR contract DAAL06-92-C-0006, to develop a computational algorithm for modeling the acoustic scattering from a turbulence distribution, which is modeled as a collection of discrete turbules with a discrete turbule size spectrum. The methods adopted here build on the structure function analysis, and incorporate this approach into a mean field method for characterizing coherent and incoherent multiple scattering from a turbulence volume. The mean field treatment is an extension to the acoustic case of prior work in electromagnetic scattering from turbulence. The object of the current effort is to produce a phase I code module that is very general, and that can, under phase II and beyond, be adapted to and incorporated in the Army's broader acoustic modeling program, with particular application to the issue of scattering into shadow zones.

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Radiation Monitoring Devices
44 Hunt Street
Watertown, Massachusetts 02172

ACOUSTIC SCATTERING BY A VORTEX MODEL OF TURBULENCE

Contract Final Report

November 19, 1992

Chris Rollins, Ph.D.
David Resendes, Ph.D.
Michael R. Squillante, Ph.D.

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Acoustic Scattering by a Vortex Model of Turbulence

Contract Final Report

Chris Rollins, Ph.D.
David Resendes, Ph.D.
Michael R. Squillante, Ph.D.

Radiation Monitoring Devices
44 Hunt Street
Watertown, Massachusetts

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Part I. Project Summary

This report summarizes the work performed under DOD SBIR contract # DAAL06-92-C-0006, to develop of a computational algorithm for modeling the acoustic scattering from a turbulence distribution, which is modeled as a collection of discrete turbules with a discrete turbule size spectrum. The methods adopted here build on the prior structure function analysis of Tatarski[1], and incorporate this approach into a mean field method for characterizing coherent and incoherent multiple scattering from a turbulence volume. The mean field treatment is an extension to the acoustic case of prior work in electromagnetic scattering from turbulence by one of the authors (Dr. Resendes).[2,3] The object of the current effort is to produce a phase I code module which is very general, and which can, under phase II and beyond, be adapted to, and incorporated in, the Army's broader acoustic modeling program, with particular application to the issue of scattering into shadow zones. This phase I SBIR project was completed on November 18, 1992.

Part II of this report addresses the structure function approach and its relationship to the computation of an appropriate eddy number density distribution as a function of eddy size parameter. The work discussed here builds upon prior work by Dr. Harry Auverman[4] and Dr. George Goedecke.[5-7] Section 1 of this part of the report includes a discussion of some of this prior work, and provides a background for the balance of the development of the structure function approach. The second section addresses the theoretical derivation of expressions for the general structure function and for the power spectral density. In section 3, we develop a refractive index structure function for a simple turbulence model, then, in section 4, we show how the minimum error method can be used to calculate a number density dependence on eddy size, for a specific eddy model. Section 5 summarizes the results of part II.

In part III of this report, we begin the development of the mean field approach to the multiple acoustic scattering problem. The self consistent field approach is applied to the multiple scattering of acoustic waves from a turbulence distribution consisting of multiple scattering sites of various sizes confined to a turbulence volume. This approach provides a general and systematic procedure for carrying out a perturbation series for the system of scatterers as a whole.

The averaging process for determining the mean field is model dependent. In the second section of part III, we present the turbulence model employed for these phase I calculations. Subsequent efforts which build from the current work may wish to employ a different or more complex turbulence model, which will require re-evaluation of the mean fields.

After describing our turbulence model, we discuss the properties of a single scatterer. This is required as an initial step in the construction of the configuration dependent equations for the scattered field of a system of N scatterers. In particular, these equations are expressed in terms of the exact single scatterer result. We also present an exact result for a particular single scatterer model. Any other model for the single site could be employed, and approximate, or even empirical solutions could be used when exact solutions are unavailable.

Following the discussion of the single scattering site, we proceed to the consideration of the coupled scattering properties of a collection of N scatterers. This section is very general, and culminates in the presentation of (1) an equation for the total wave in terms of its contributions from the incident wave and the scattered wave from each of the N sites, and (2) a system of self consistent coupled equations satisfied by the set of scattered waves from the N sites. These equations, taken together, fully describe the multiple scattering problem.

Following the development of the equations for the scattering from N sites, we briefly discuss the probabilistic and statistical aspects involved in the characterization of the configuration of the N scatterers. We then proceed to the evaluation of the coherent and the incoherent scattered fields, as expressed in terms of the mean field. We next present the results of the preceding sections in a plane wave basis. Evaluation of the acoustic signal which would be received at a remote detector is accomplished by means of the far field approximation to the Kirchoff integral.

At the end of the theoretical development, a comparison is made between the current formalism and the "first Born" treatment of Tatarski. It is shown that Tatarski's calculation of the scattered power follows from the more general treatment employed here by keeping lowest order terms only, and making further simplifying assumptions in the averaging process.

In the final sections of the effort, we present the numerical arguments employed in the computational algorithm, as well as provide the text of the code itself. We also discuss the ways in which this code can be incorporated into the shadow zone analysis.

PART II

Eddy Number Density Distribution and the Structure Function

Introduction:

A direct calculational approach for the eddy number density distribution is developed, using the formalisms of the index of refraction structure function, and the minimum error method for calculating number density distribution for a particular model of turbulence. Much of the work presented here follows closely the form and content of prior analyses by Dr. G. Goedecke, from which it in part derives its motivation. Significant departures are made from Dr. Goedecke's work in that assumptions that eddies must be confined to a lattice-like array of cells which subdivide the volume are dropped in favor of the presumption that any given eddy may be found anywhere in the entire turbulence volume with equal probability. We also depart from Dr. Goedecke by choosing to adopt an approximate means to obtaining the number density distribution from a model which has been fitted parametrically to the theoretical power spectral density, rather than constraining ourselves to accepting only a model from which the ideal power spectral density itself can be derived. These departures permit us somewhat greater latitude in the development of an application oriented algorithm for modeling acoustic scattering effects for a turbulent atmosphere. We begin the discussion by addressing the latter departure.

One might well expect to be able to derive the structure function and the corresponding power spectral density of the fluctuating refractive index in terms of the constituent eddies which make up the turbulence volume which exhibits these fluctuations. There is, however, a subtlety here, related to the theoretical status of the structure function and of the turbulent eddies. While at least in the inertial subrange, the structure function is derived from physical laws and principles, a superposition of eddies of differing size is a highly simplified and idealized *model* of turbulence. It is therefore unlikely that the true physical structure function should be *derivable* from a particular model collection of eddies. A more modest expectation might be that an appropriate collection of eddies should allow one to approximate, to the desired accuracy, the physical structure function.

Mathematically, this observation on the non-derivability of the structure function from a simplified model presents itself in the fact that it is not possible to obtain a particular arbitrary function, such as $f \sim r^{2/3}$, from the superposition of an incomplete set of functions, such as those which are discussed below in the context of the power spectral density.

Lacking closure for the basis functions, the Minimum Error Method provides a means for generating an approximate representation of the structure function in terms of an eddy superposition. We detail below a method for obtaining a suitable structure function approximation. The general outline is as follows:

We begin by making some observations on prior work. Next we write down the general theoretical expressions for a general structure function and a general power spectral density. This is followed by a development of expressions for these quantities based on an eddy model of turbulence. The Minimum Error procedure is then used to determine the optimal eddy number density distribution. Finally, we present a brief summary.

Section II.1

Observations on Prior Work

As discussed above, the work conducted in this program is rooted in the prior efforts of Dr. Harry Auverman and Dr. George Goedecke. This prior work, both at and sponsored by the Meteorology and Acoustics branch of the Army Atmospheric Sciences Laboratory, emphasizes the development of a reasonable physical characterization of acoustic scattering from turbulence, using a model built from the superposition of constituent eddies. During the course of review of this earlier work, it became apparent that certain aspects of the formalism might impose more constraints than were completely necessary for the development of an acceptable scattering model. This section addresses these difficulties.

The general formalism which was earlier developed seeks to first characterize the number density of constituent eddies as a function of eddy size and then, for each size, constructs a lattice of cells within the turbulence volume such that each cell contains exactly one eddy and the collection of cells for a given size yields the correct volumetric eddy number density. This procedure ensures that the correct number density versus eddy size distribution is achieved throughout the entire turbulence volume. Each eddy is presumed to exhibit an index of refraction fluctuation which is localized in space by an envelope function which decays to zero in some manner as the distance from the eddy center increases. Examples of such a construct would include a Gaussian envelope function, which decays asymptotically to zero at large radius, or a "hard sphere" model in which

the index fluctuation falls abruptly to zero at some particular radius. In either case, the eddy is characterized by some particular size.

In addition to the spatial localization of the refractive index fluctuation about a centroid, the cell approach involves the localization of the centroid of the eddy within the cell volume. While this aspect of the model leads to a computationally reasonable description of the system of eddies , it appears to lack physical motivation, since it constrains the eddies to particular regions of space without providing any particular physical force to implement this constraint. It is apparent, for example, that a large longitudinal density fluctuation in the overall medium could, in the physical world, result in the near total depletion of eddies of a given size over a scale larger than the cell dimension. Such depletion is, however, specifically denied by the cell formalism.

In addition to this physical difficulty, there is also a computational difficulty with the prior treatment. This difficulty arises as a consequence of the factorization of the averaging procedure over the turbule distribution. The ability to factor an ensemble average into the product of two separate averages over sub-ensembles arises in the event that the two sub-ensembles are uncorrelated. In the earlier work, an initial factorization is done into separate ensembles for eddies of differing size. A subsequent factorization also separates the average into the product of separate averages for the density fluctuation at different sites for eddies of a single size.

This particular factorization is a relatively commonplace practice in the computation of multiple scattering effects, and is done in an effort to obtain a sufficiently simple description of the system that one might be able to find a solution, but the assumptions carry with them a secondary consequence. The factorization is only possible if the eddies are uncorrelated, since otherwise the variables are not independent. This particular factorization is therefore equivalent to stating that no two eddies communicate in any way, or similarly, that the eddies are entirely uncorrelated. If the eddies are uncorrelated, then once again there is no longer any particular reason to believe that they should be confined to a particular region of space surrounding a lattice point. One would be more inclined to believe that any particular eddy has an equal probability of being found anywhere within the entirety of the turbulence volume. In this case, the probability function which describes the likelihood of the eddy being at a particular point in space would be constant throughout space. The corresponding Fourier transform of this spatial probability distribution would then be a delta function in k space.

One final observation which was made during the review of the earlier work has to do with the small r dependence of the derived structure function. The structure function can be rather generally described as being of the form

$$D_f(r) = \int dk\, \Phi(k)\, (1 - \frac{\sin(kr)}{kr}).$$

A derivation of this form and a discussion of the assumptions which lead to this particular form are presented below. This function has a quadratic small r dependence, provided only that there is a maximum value of k for which $\Phi(k)$ is non zero, owing to the small angle properties of the sine function. If $\Phi(k)$ is zero for all k larger than about $2\pi/\ell$, where ℓ is the small end of the inertial subrange, then all sine contributions to D will be in the small argument limit for $r<\ell$, and the function will exhibit quadratic behavior. A consequence of this observation is that it will not be necessary to compute a particular $\Phi(k)$ associated with the small r region of D.

Section II.2

Expressions Derived from Theory for the Structure Function and the Power Spectral Density

In analyzing the spatial structure of meteorological fields, it is appropriate to apply the method of structure functions. The concept of structure function arises from a consideration of a random function whose mean value changes slowly and smoothly, rather than remaining constant.

A difficulty which arises in the analysis of such functions is that it is not immediately obvious where to define the boundary between true variation of the mean value and very slow fluctuations about that mean. Whenever the mean value $\langle f \rangle$ changes, we can consider instead the difference $F = f(\vec{r}) - f(\vec{r} + \vec{R})$. For values of $\vec{R}$ which are not too large, slow changes in f do not affect this difference (at least approximately).
The structure function is defined by

$$D_f(\vec{r}_1, \vec{r}_2) \equiv \langle\, [f(\vec{r}_1) - f(\vec{r}_2)]^2 \,\rangle \,. \qquad \text{II.2.1}$$

The difference between the values of $f(\vec{r})$ at two points $\vec{r}_1$ and $\vec{r}_2$ is chiefly affected only by inhomogeneities of the field with dimensions which do not exceed the distance $|\vec{r}_1 - \vec{r}_2|$. If this distance is not too large, then the largest anisotropic inhomogeneities have no effect on $f(\vec{r}_1) - f(\vec{r}_2)$, and $D_f(\vec{r}_1, \vec{r}_2)$ can depend only on $(\vec{r}_1 - \vec{r}_2)$. By contrast, the correlation function $\mu_f(\vec{r}_1, \vec{r}_2)$, defined by

$$\mu_f(\vec{r}_1, \vec{r}_2) \equiv \langle \left[f(\vec{r}_1) - \langle f(\vec{r}_1) \rangle \right] \left[f(\vec{r}_2) - \langle f(\vec{r}_2) \rangle \right] \rangle , \qquad \text{II.2.2}$$

is affected by inhomogeneities of all scales. In addition to this homogeneity property, if the random field is also isotropic, then the structure function depends only on $| \vec{r}_1 - \vec{r}_2 |$.

A locally homogeneous random field $f(\vec{r})$ can be represented by

$$f(\vec{r}) = f(\vec{0}) + \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (1 - e^{i \vec{k} \cdot \vec{r}}) \, d\Phi(\vec{k}), \qquad \text{II.2.3}$$

where

$$\overline{d\Phi(\vec{k}_1) \; d\Phi^*(\vec{k}_2)} = \delta(\vec{k}_1 - \vec{k}_2) \, \Phi(\vec{k}_1) \, d\vec{k}_1 \, d\vec{k}_2 \qquad \text{II.2.4}$$

and $\Phi(\vec{k}) \geq 0$ is the spectral density of f. A general form for the structure function can be obtained for the locally homogeneous random field by substituting the general form of the field (equation II.2.3) into the definition of the structure function (equation II.2.1), and evaluating the average, making use of the relation II.2.4. The result is:

$$D_f(\vec{r}_1, \vec{r}_2) = D_f(\vec{r}_1 - \vec{r}_2, \vec{0}) \equiv D_f(\vec{r}, \vec{0}) \equiv D_f(\vec{r})$$

$$= \langle [f(\vec{r}) - f(\vec{0})]^2 \rangle$$

$$= \langle [f(\vec{r}) - f(\vec{0})] \, [f(\vec{r}) - f(\vec{0})]^* \rangle \qquad \text{(f is real)}$$

$$= \left\langle \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (1 - e^{i \vec{k}_1 \cdot \vec{r}}) \, d\Phi(\vec{k}_1) \; \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (1 - e^{i \vec{k}_2 \cdot \vec{r}}) \, d\Phi(\vec{k}_2) \right\rangle$$

$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} d\vec{k}_1 \, \Phi(\vec{k}_1) \; | (1 - e^{i \vec{k}_1 \cdot \vec{r}}) |^2$$

$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} d\vec{k}_1 \, \Phi(\vec{k}_1) \; (1 - \cos \vec{k}_1 \cdot \vec{r}) \qquad \text{II.2.5}$$

If, in addition, f is locally isotropic, then $D_f(\vec{r}) = D_f(r)$ and $\Phi(\vec{k}) = \Phi(k)$, and relation II.2.5 can be further simplified. In particular,

$$D_f(r) = 2 \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} k^2 \, dk \sin\theta \, d\theta \, d\phi \, \Phi(k) \, [1 - \cos(kr \cos\theta)], \qquad \text{II.2.6}$$

where θ is the angle between $\vec{k}$ and $\vec{r}$. On performing the integrations, this gives

$$D_f(r) = 8\pi \int_0^{\infty} \left(1 - \frac{\sin kr}{kr} \right) \Phi(k) \, k^2 \, dk. \qquad \text{II.2.7}$$

We therefore come to the overall conclusion that for a locally homogeneous, locally isotropic random field, the structure function is of the form II.2.7. As discussed above, this function has a quadratic small r dependence if $\Phi(k)$ posesses a large k cut off, independent of the detailed character of the spectral density function Φ. In the next section, we apply the formalism of this section to an eddy model of turbulence, to characterize the acoustic response of the system.

Section II.3.

Refractive Index Structure Function for a Simple Turbulence Model

As a model of turbulence we consider a collection of localized spherical eddies whose size parameter forms a discrete spectrum. It is assumed that this description can be made arbitrarily close to a continuum size distribution by increasing the number of sizes allowed in the discrete size spectrum. We denote the number of sizes as N_S, and the number of eddies of each size by N_α. The distribution of eddies is assumed to be completely random within a cubical volume $V = L^3$. It assumed that the individual eddies may each be found anywhere in the volume V.

The refractive index for this turbulence model can be taken to be a uniform background value, assumed to be unity, with a superposed fluctuation about this background value associated with each of the eddies. We assume that all eddies of the same size have the same functional form for their associated localized refractive index fluctuation. For the chosen decomposition of the eddy population, this statement can be written as

$$n(\vec{r}) - 1 = \sum_{\alpha=1}^{N_S} \sum_{n_\alpha=1}^{N_\alpha} f_\alpha(\vec{r} - \vec{r}_{n_\alpha}), \qquad \text{II.3.1}$$

where $n(\vec{r})$ is the refractive index of the turbulent system, $\vec{r}_{n_\alpha}$ is the position of eddy number n_α of size parameter α, and f_α is the function which describes the localized refractive index fluctuation associated with an eddy of size parameter α.

The correlation function for this refractive index model is given by

$$\mu(\vec{r}\,,\,\vec{r}') \equiv \langle (n(\vec{r}) - 1)\,(n(\vec{r}') - 1)\rangle \qquad \text{(generally)}$$

$$= \sum_{\alpha=1}^{N_s} \sum_{\beta=1}^{N_s} \sum_{n_\alpha=1}^{N_\alpha} \sum_{m_\beta=1}^{N_\beta} \langle\, f_\alpha(\vec{r} - \vec{r}_{n_\alpha})\, f_\beta(\vec{r}' - \vec{r}_{m_\beta})\rangle\,. \qquad \text{II.3.2}$$

The structure function may be expressed in terms of the correlation function as

$$D_n(\vec{r}\,,\,\vec{r}') \equiv \langle [\,(n(\vec{r}) - 1) - (n(\vec{r}') - 1)\,]^2\rangle$$

$$= \mu(\vec{r}\,,\,\vec{r}\,) + \mu(\vec{r}',\vec{r}') - 2\mu(\vec{r}\,,\,\vec{r}') \qquad \text{II.3.3}$$

To evaluate the structure function for the selected eddy model, we first evaluate the correlation function. It is convenient to separate relation II.3.2 into two terms. The first contribution is that portion which is diagonal in the size parameter, i.e.. for which $\beta = \alpha$. The second contribution is from the off diagonal terms. This yields

$$\mu(\vec{r}\,,\,\vec{r}') = \sum_{\alpha=1}^{N_s} \sum_{n_\alpha=1}^{N_\alpha} \sum_{m_\alpha=1}^{N_\alpha} \langle\, f_\alpha(\vec{r} - \vec{r}_{n_\alpha})\, f_\alpha(\vec{r}' - \vec{r}_{m_\alpha})\rangle$$

$$+ \sum_{\alpha=1}^{N_s} \sum_{\substack{\beta=1 \\ (\beta\neq\alpha)}}^{N_s} \sum_{n_\alpha=1}^{N_\alpha} \sum_{m_\beta=1}^{N_\beta} \langle\, f_\alpha(\vec{r} - \vec{r}_{n_\alpha})\, f_\beta(\vec{r}' - \vec{r}_{m_\beta})\rangle\,. \qquad \text{II.3.4}$$

A similar separation may be conducted for the eddy indices n and m, for a given size parameter. This yields

$$\mu(\vec{r}\,,\,\vec{r}') = \sum_{\alpha=1}^{N_s} \sum_{n_\alpha=1}^{N_\alpha} \langle\, f_\alpha(\vec{r} - \vec{r}_{n_\alpha})\, f_\alpha(\vec{r}' - \vec{r}_{n_\alpha})\rangle$$

$$+ \sum_{\alpha=1}^{N_s} \sum_{n_\alpha=1}^{N_\alpha} \sum_{\substack{m_\alpha=1 \\ (n_\alpha\neq m_\alpha)}}^{N_\alpha} \langle\, f_\alpha(\vec{r} - \vec{r}_{n_\alpha})\, f_\alpha(\vec{r}' - \vec{r}_{m_\alpha})\rangle$$

$$+ \sum_{\alpha=1}^{N_s} \sum_{\substack{\beta=1 \\ (\beta \neq \alpha)}}^{N_s} \sum_{n_\alpha=1}^{N_\alpha} \sum_{m_\beta=1}^{N_\beta} \langle\, f_\alpha(\vec{r} - \vec{r}_{n_\alpha})\, f_\beta(\vec{r}' - \vec{r}_{m_\beta}) \rangle \,. \qquad \text{II.3.5}$$

At this point, it is appropriate to begin making simplifying assumptions. We will follow here the same approach as that taken by Dr. Goedecke and others. The first assumption we will make is that averages over turbules of differing size are uncorrelated. This will allow factorization of the last term in relation II.3.5. The result is

$$\mu(\vec{r}\,,\,\vec{r}') = \sum_{\alpha=1}^{N_s} \sum_{n_\alpha=1}^{N_\alpha} \langle\, f_\alpha(\vec{r} - \vec{r}_{n_\alpha})\, f_\alpha(\vec{r}' - \vec{r}_{n_\alpha}) \rangle$$

$$+ \sum_{\alpha=1}^{N_s} \sum_{n_\alpha=1}^{N_\alpha} \sum_{\substack{m_\alpha=1 \\ (n_\alpha \neq m_\alpha)}}^{N_\alpha} \langle\, f_\alpha(\vec{r} - \vec{r}_{n_\alpha})\, f_\alpha(\vec{r}' - \vec{r}_{m_\alpha}) \rangle$$

$$+ \sum_{\alpha=1}^{N_s} \sum_{\substack{\beta=1 \\ (\beta \neq \alpha)}}^{N_s} \sum_{n_\alpha=1}^{N_\alpha} \sum_{m_\beta=1}^{N_\beta} \langle\, f_\alpha(\vec{r} - \vec{r}_{n_\alpha}) \,\rangle \langle\, f_\beta(\vec{r}' - \vec{r}_{m_\beta}) \rangle \,.$$

We next make the further simplifying assumption that averages over different sites, even of the same size, are uncorrelated. This gives

$$\mu(\vec{r}\,,\,\vec{r}') = \sum_{\alpha=1}^{N_s} \sum_{n_\alpha=1}^{N_\alpha} \langle\, f_\alpha(\vec{r} - \vec{r}_{n_\alpha})\, f_\alpha(\vec{r}' - \vec{r}_{n_\alpha}) \rangle$$

$$+ \sum_{\alpha=1}^{N_s} \sum_{n_\alpha=1}^{N_\alpha} \sum_{\substack{m_\alpha=1 \\ (m_\alpha \neq n_\alpha)}}^{N_\alpha} \langle\, f_\alpha(\vec{r} - \vec{r}_{n_\alpha}) \,\rangle \langle\, f_\alpha(\vec{r}' - \vec{r}_{m_\alpha}) \rangle$$

$$+ \sum_{\alpha=1}^{N_s} \sum_{\substack{\beta=1 \\ (\beta \neq \alpha)}}^{N_s} \sum_{n_\alpha=1}^{N_\alpha} \sum_{m_\beta=1}^{N_\beta} \langle\, f_\alpha(\vec{r} - \vec{r}_{n_\alpha}) \,\rangle \langle\, f_\beta(\vec{r}' - \vec{r}_{m_\beta}) \rangle .$$

It should be pointed out here that these particular assumptions amount to saying that the likelihood of finding a particular refractive index fluctuation at any given place is completely unrelated to the likelihood of finding any other such fluctuation at any other place. This situation is really not very likely, and these assumptions are in all probability violated. Nevertheless, we make these assumptions anyway, in an effort to obtain a base solution from which to build a more sophisticated description.

We continue with the development of the equations, by adding and subtracting like terms so as to complete the sums:

$$\mu(\vec{r}\,,\,\vec{r}') = \sum_{\alpha=1}^{N_s} \sum_{n_\alpha=1}^{N_\alpha} \langle\, f_\alpha(\vec{r} - \vec{r}_{n_\alpha})\, f_\alpha(\vec{r}' - \vec{r}_{n_\alpha}) \rangle$$

$$+ \sum_{\alpha=1}^{N_s} \sum_{n_\alpha=1}^{N_\alpha} \sum_{m_\alpha=1}^{N_\alpha} \langle\, f_\alpha(\vec{r} - \vec{r}_{n_\alpha}) \,\rangle \langle\, f_\alpha(\vec{r}' - \vec{r}_{m_\alpha}) \rangle$$

$$- \sum_{\alpha=1}^{N_s} \sum_{n_\alpha=1}^{N_\alpha} \langle\, f_\alpha(\vec{r} - \vec{r}_{n_\alpha}) \rangle \langle\, f_\alpha(\vec{r}' - \vec{r}_{n_\alpha}) \rangle$$

$$+ \sum_{\alpha=1}^{N_s} \sum_{\beta=1}^{N_s} \sum_{n_\alpha=1}^{N_\alpha} \sum_{m_\beta=1}^{N_\beta} \langle\, f_\alpha(\vec{r} - \vec{r}_{n_\alpha}) \,\rangle \langle\, f_\beta(\vec{r}' - \vec{r}_{m_\beta}) \rangle$$

$$- \sum_{\alpha=1}^{N_s} \sum_{n_\alpha=1}^{N_\alpha} \sum_{m_\alpha=1}^{N_\alpha} \langle f_\alpha(\vec{r} - \vec{r}_{n_\alpha}) \rangle \langle f_\alpha(\vec{r}' - \vec{r}_{m_\alpha}) \rangle .$$

Which can be rewritten as

$$\mu(\vec{r}, \vec{r}') = \sum_{\alpha=1}^{N_s} \sum_{n_\alpha=1}^{N_\alpha} \langle f_\alpha(\vec{r} - \vec{r}_{n_\alpha}) f_\alpha(\vec{r}' - \vec{r}_{n_\alpha}) \rangle$$

$$+ \sum_{\alpha=1}^{N_s} \left[\sum_{n_\alpha=1}^{N_\alpha} \langle f_\alpha(\vec{r} - \vec{r}_{n_\alpha}) \rangle \right] \left[\sum_{m_\alpha=1}^{N_\alpha} \langle f_\alpha(\vec{r}' - \vec{r}_{m_\alpha}) \rangle \right]$$

$$- \sum_{\alpha=1}^{N_s} \sum_{n_\alpha=1}^{N_\alpha} \left[\langle f_\alpha(\vec{r} - \vec{r}_{n_\alpha}) \rangle \right] \left[\langle f_\alpha(\vec{r}' - \vec{r}_{n_\alpha}) \rangle \right]$$

$$+ \sum_{\alpha=1}^{N_s} \sum_{n_\alpha=1}^{N_\alpha} \left[\langle f_\alpha(\vec{r} - \vec{r}_{n_\alpha}) \rangle \right] \left[\sum_{\beta=1}^{N_s} \sum_{m_\beta=1}^{N_\beta} \langle f_\beta(\vec{r}' - \vec{r}_{m_\beta}) \rangle \right]$$

$$- \sum_{\alpha=1}^{N_s} \left[\sum_{n_\alpha=1}^{N_\alpha} \langle f_\alpha(\vec{r} - \vec{r}_{n_\alpha}) \rangle \right] \left[\sum_{m_\alpha=1}^{N_\alpha} \langle f_\alpha(\vec{r}' - \vec{r}_{m_\alpha}) \rangle \right].$$

Next we simplify the notation by introducing the functions

$$S_\alpha(\vec{r}) \equiv \sum_{n_\alpha=1}^{N_\alpha} \langle f_\alpha(\vec{r} - \vec{r}_{n_\alpha}) \rangle$$

$$S(\vec{r}) \equiv \sum_{\alpha=1}^{N_s} S_\alpha(\vec{r}),$$

which yields

$$\mu(\vec{r}, \vec{r}') = \sum_{\alpha=1}^{N_s} \sum_{n_\alpha=1}^{N_\alpha} \langle f_\alpha(\vec{r} - \vec{r}_{n_\alpha}) f_\alpha(\vec{r}' - \vec{r}_{n_\alpha}) \rangle$$

$$+ \sum_{\alpha=1}^{N_s} [S_\alpha(\vec{r})][S_\alpha(\vec{r}')]$$

$$- \sum_{\alpha=1}^{N_s} \sum_{n_\alpha=1}^{N_\alpha} \langle f_\alpha(\vec{r} - \vec{r}_{n_\alpha}) \rangle \langle f_\alpha(\vec{r}' - \vec{r}_{n_\alpha}) \rangle$$

$$+ S(\vec{r}) S(\vec{r}')$$

$$- \sum_{\alpha=1}^{N_s} [S_\alpha(\vec{r})][S_\alpha(\vec{r}')],$$

which simplifies to

$$\mu(\vec{r}, \vec{r}') =$$

$$\sum_{\alpha=1}^{N_s} \sum_{n_\alpha=1}^{N_\alpha} [\langle f_\alpha(\vec{r} - \vec{r}_{n_\alpha}) f_\alpha(\vec{r}' - \vec{r}_{n_\alpha}) \rangle - \langle f_\alpha(\vec{r} - \vec{r}_{n_\alpha}) \rangle \langle f_\alpha(\vec{r}' - \vec{r}_{n_\alpha}) \rangle]$$

$$+ S(\vec{r}) S(\vec{r}') . \qquad \text{II.3.6}$$

The corresponding results for $\mu(\vec{r}, \vec{r})$ and $\mu(\vec{r}', \vec{r}')$ are obtained by direct substitution, and the results entered into equation II.3.3, to obtain the structure function $D(\vec{r}, \vec{r}')$.

In order to evaluate the averages in the above equations, it is convenient to introduce a Fourier series representation. The plane wave basis described by the set of functions

$$\left\{ \frac{e^{i\left(\frac{2n\pi x}{L}\right)}}{\sqrt{L}} \right\} , n=0, \pm 1, \pm 2, \pm 3... \qquad \text{II.3.7}$$

is a complete orthonormal set on the interval $[-\frac{L}{2}, \frac{L}{2}]$. We may therefore expand any function of $\vec{r}$ defined in the volume $V = L^3$ according to

$$f(\vec{r}) = \sum_{k_x=-\infty}^{+\infty} \sum_{k_y=-\infty}^{+\infty} \sum_{k_z=-\infty}^{+\infty} \tilde{f}_{\vec{k}} \; e^{i\left(\vec{k} \cdot \vec{r}\right)}$$
$$\equiv \sum_{\vec{k}} \tilde{f}_{\vec{k}} \; e^{i\left(\vec{k} \cdot \vec{r}\right)} , \qquad \text{II.3.8}$$

where

$$\tilde{f}_{\vec{k}} = \frac{1}{V} \int_V d^3\vec{r} \; f(\vec{r}) \; e^{-i\,\vec{k} \cdot \vec{r}} , \qquad \text{II.3.9}$$

and

$$\vec{k} = \frac{2\pi}{L} \vec{\ell} \quad ; \quad \vec{\ell} = (\ell_1, \ell_2, \ell_3) \; ; \; -\infty < \ell_i < \infty, \; \ell_i \in \{\text{integers}\}$$

For this basis set, the completeness and orthonormality are respectively:

$$\frac{1}{V}\sum_{\vec{k}} e^{i\vec{k}\cdot(\vec{r}-\vec{r}')} = \delta(\vec{r}-\vec{r}') \qquad \text{II.3.10}$$

and

$$\frac{1}{V}\int_V d^3\vec{r}\; e^{-i(\vec{k}-\vec{k}')\cdot\vec{r}} = \delta_{\vec{k},\vec{k}'} \,. \qquad \text{II.3.11}$$

Making use of these last few results, it is now possible to evaluate the averages in the functions $D(\vec{r},\vec{r}')$ and $\mu(\vec{r},\vec{r}')$. The average of the function $f_\alpha(\vec{r}-\vec{r}_{n_\alpha})$ is given by

$$\langle f_\alpha(\vec{r}-\vec{r}_{n_\alpha})\rangle = \int_V d^3\vec{r}_{n_\alpha}\; f_\alpha(\vec{r}-\vec{r}_{n_\alpha})\, p_\alpha(\vec{r}_{n_\alpha}),$$

where $p_\alpha(\vec{r}_{n_\alpha})$ is the probability that the eddy n_α is located at the position $\vec{r}_{n_\alpha}$. Making use of the result II.3.8, this can be rewritten as

$$\langle f_\alpha(\vec{r}-\vec{r}_{n_\alpha})\rangle = \int_V d^3\vec{r}_{n_\alpha} \sum_{\vec{k}} \tilde{f}_{\alpha\vec{k}}\; e^{i\vec{k}\cdot(\vec{r}-\vec{r}_{n_\alpha})}\; p_\alpha(\vec{r}_{n_\alpha})$$

$$= \sum_{\vec{k}} \tilde{f}_{\alpha\vec{k}}\; e^{i\vec{k}\cdot\vec{r}} \int_V d^3\vec{r}_{n_\alpha}\; e^{-i\vec{k}\cdot\vec{r}_{n_\alpha}}\; p_\alpha(\vec{r}_{n_\alpha}). \qquad \text{II.3.12}$$

At this point, we make our departure from Dr. Goedecke's treatment. We will make the assumption that the eddies are not confined to localized cells, but rather are free to wander anywhere in the volume V with equal probability. This assumption can be expressed as

$$p_\alpha(\vec{r}_{n_\alpha}) = \frac{1}{V} \,. \qquad \text{II.3.13}$$

Substituting this into the integral, we obtain

$$\langle f_\alpha(\vec{r}-\vec{r}_{n_\alpha})\rangle = \sum_{\vec{k}} \tilde{f}_{\alpha\vec{k}}\; e^{i\vec{k}\cdot\vec{r}}\; \frac{1}{V}\int_V d^3\vec{r}_{n_\alpha}\; e^{-i\vec{k}\cdot\vec{r}_{n_\alpha}} \,.$$

$$= \sum_{\vec{k}} \tilde{f}_{\alpha\vec{k}} \; e^{i\vec{k}\cdot\vec{r}} \; \delta_{\vec{k},\vec{0}}$$

$$= \tilde{f}_{\alpha\vec{0}} \qquad \text{II.3.14}$$

Similarly, we find

$$\langle f_\alpha(\vec{r} - \vec{r}_{n_\alpha}) \rangle \langle f_\alpha(\vec{r}' - \vec{r}_{m_\alpha}) \rangle$$

$$= \langle \sum_{\vec{k}} \tilde{f}_{\alpha\vec{k}} \; e^{i\vec{k}\cdot(\vec{r} - \vec{r}_{n_\alpha})} \sum_{\vec{k}'} \tilde{f}_{\alpha\vec{k}'} \; e^{i\vec{k}'\cdot(\vec{r}' - \vec{r}_{n_\alpha})} \rangle$$

$$= \sum_{\vec{k}} \sum_{\vec{k}'} \tilde{f}_{\alpha\vec{k}} \, \tilde{f}_{\alpha\vec{k}'} \, e^{i\vec{k}\cdot\vec{r}} \, e^{i\vec{k}'\cdot\vec{r}'} \langle e^{-i(\vec{k} + \vec{k}')\cdot\vec{r}_{n_\alpha}} \rangle$$

$$= \sum_{\vec{k}} \tilde{f}_{\alpha\vec{k}} \, \tilde{f}_{\alpha-\vec{k}} \; e^{i\vec{k}\cdot(\vec{r} - \vec{r}')}$$

$$= \sum_{\vec{k}} \tilde{f}_{\alpha\vec{k}} \, \tilde{f}_{\alpha-\vec{k}} \; e^{i\vec{k}\cdot\vec{R}} \qquad \text{II.3.15}$$

Substituting II.3.14 and II.3.15 into II.3.6, we find

$$\mu(\vec{r}, \vec{r}') = \sum_{\alpha=1}^{N_s} \sum_{n_\alpha=1}^{N_\alpha} \Big[\sum_{\vec{k}} \tilde{f}_{\alpha\vec{k}} \, \tilde{f}_{\alpha-\vec{k}} \; e^{i\vec{k}\cdot\vec{R}} - \tilde{f}_{\alpha\vec{0}} \, \tilde{f}_{\alpha\vec{0}} \Big]$$

$$+ \sum_{\alpha=1}^{N_s} \sum_{n_\alpha=1}^{N_\alpha} \tilde{f}_{\alpha\vec{0}} \sum_{\beta=1}^{N_s} \sum_{n_\beta=1}^{N_\beta} \tilde{f}_{\beta\vec{0}}$$

$$= \sum_{\alpha=1}^{N_s} N_\alpha \Big[\sum_{\vec{k}} \tilde{f}_{\alpha\vec{k}} \, \tilde{f}_{\alpha-\vec{k}} e^{i\vec{k}\cdot\vec{R}} - (\tilde{f}_{\alpha\vec{0}})^2 \Big] + \Big(\sum_{\alpha=1}^{N_s} N_\alpha \, \tilde{f}_{\alpha\vec{0}}\Big)^2$$

$$= \sum_{\alpha=1}^{N_s} N_\alpha \sum_{\vec{k}} \tilde{f}_{\alpha\vec{k}}\, \tilde{f}_{\alpha,-\vec{k}}\, e^{i\vec{k}\cdot\vec{R}} - \sum_{\alpha=1}^{N_s} N_\alpha (\tilde{f}_{\alpha\vec{0}})^2$$

$$+ (\sum_{\alpha=1}^{N_s} N_\alpha\, \tilde{f}_{\alpha\vec{0}})^2 . \qquad \text{II.3.16}$$

Similarly, we have

$$\mu(\vec{r},\vec{r}) = \mu(\vec{r}',\vec{r}') = \sum_{\alpha=1}^{N_s} N_\alpha \sum_{\vec{k}} \tilde{f}_{\alpha\vec{k}}\, \tilde{f}_{\alpha,-\vec{k}} - \sum_{\alpha=1}^{N_s} N_\alpha (\tilde{f}_{\alpha\vec{0}})^2 + (\sum_{\alpha=1}^{N_s} N_\alpha\, \tilde{f}_{\alpha\vec{0}})^2 .$$

II.3.17

Substitution of the correlation function into equation II.3.3 for the structure function, gives

$$D_n(\vec{r},\vec{r}') = - 2\sum_{\alpha=1}^{N_s} N_\alpha \sum_{\vec{k}} \tilde{f}_{\alpha\vec{k}}\, \tilde{f}_{\alpha,-\vec{k}}$$

$$+ 2\sum_{\alpha=1}^{N_s} N_\alpha (\tilde{f}_{\alpha\vec{0}})^2 - 2(\sum_{\alpha=1}^{N_s} N_\alpha\, \tilde{f}_{\alpha\vec{0}})^2 .$$

$$+ 2\sum_{\alpha=1}^{N_s} N_\alpha \sum_{\vec{k}} \tilde{f}_{\alpha\vec{k}}\, \tilde{f}_{\alpha,-\vec{k}} e^{i\vec{k}\cdot\vec{R}}$$

$$- 2\sum_{\alpha=1}^{N_s} N_\alpha (\tilde{f}_{\alpha\vec{0}})^2 + 2(\sum_{\alpha=1}^{N_s} N_\alpha\, \tilde{f}_{\alpha\vec{0}})^2$$

$$= 2\sum_{\alpha=1}^{N_s} N_\alpha \sum_{\vec{k}} \tilde{f}_{\alpha\vec{k}}\, \tilde{f}_{\alpha,-\vec{k}} (1 - e^{i\vec{k}\cdot\vec{R}}), \qquad \text{II.3.18}$$

which can also be written as

$$D_n(\vec{r},\vec{r}') = 2\sum_{\alpha=1}^{N_s} N_\alpha \sum_{\vec{k}} \tilde{f}_{\alpha\vec{k}}\ \tilde{f}_{\alpha\,-\vec{k}}\ (1-\cos\vec{k}\cdot\vec{R}), \qquad \text{II.3.19}$$

because the sum is over all k_x, k_y, k_z from $-\infty$ to $+\infty$, and $(\tilde{f}_{\alpha\vec{k}}\ \tilde{f}_{\alpha\,-\vec{k}})$ is even.

If we now take the limit as $L \to \infty$, the result is $\sum_{\vec{k}} \to \frac{V}{(2\pi)^3}\int d^3\vec{k}$, which yields

$$D_n(\vec{r},\vec{r}') = 2\sum_{\alpha=1}^{N_s} N_\alpha \frac{V}{(2\pi)^3}\int d^3\vec{k}\ \tilde{f}_{\alpha\vec{k}}\ \tilde{f}_{\alpha\,-\vec{k}}\ (1-\cos\vec{k}\cdot\vec{R}), \qquad \text{II.3.20}$$

Which is of the same form as Tatarski equation. 1.41:

$$D_n(\vec{r},\vec{r}') = 2\int d^3\vec{k}\ \Phi(\vec{k})\,(1-\cos\vec{k}\cdot\vec{R}),$$

with

$$\Phi(\vec{k}) = \frac{V}{(2\pi)^3}\sum_{\alpha=1}^{N_s} N_\alpha\ \tilde{f}_{\alpha\vec{k}}\ \tilde{f}_{\alpha\,-\vec{k}}\,. \qquad \text{II.3.21}$$

section II.4

Minimum Error Method

In this section we apply the minimum error method[8,9] to determine the optimal number density distribution corresponding to a specified theoretical power spectral density function. The total squared error of the model power spectral density $\Phi(k)_{model}$ relative to a theoretical power spectral density $\Phi(k)_{theory}$ is given by

$$E(\Phi(k)_{model}) = \int_{k_{min}}^{k_{max}} [\Phi(k)_{theory} - \Phi(k)_{model}]^2 dk \qquad \text{II.4.1}$$

In relation II.4.1, $\Phi(k)_{theory}$ is derived from the structure function, while $\Phi(k)_{model}$ may be expressed as a sum of the form

$$\Phi(k)_{model} = \sum_{\alpha=1}^{N_s} n_\alpha \Psi_\alpha , \qquad \text{II.4.2}$$

where the n_α are the number densities of eddies of size parameter α, which serve as expansion coefficients to be optimized. The Ψ_α are particular to the model, and will be discussed further below.

The quantity $E(\Phi(k)_{model})$ is minimized if

$$\frac{\partial E}{\partial n_\beta} = 0, \quad \text{for } \beta = 1, 2, ..., N_s. \qquad \text{II.4.3}$$

Performing the differentiations in II.4.3 yields

$$0 = \frac{\partial E}{\partial n_\beta} = \int_{k_{min}}^{k_{max}} dk \; 2\left[\Phi(k)_{theory} - \Phi(k)_{model}\right] \frac{\partial}{\partial n_\beta} \Phi(k)_{model} .$$

On evaluating $\frac{\partial}{\partial n_\beta} \Phi(k)_{model}$, the result is

$$\frac{\partial}{\partial n_\beta} \Phi(k)_{model} = \frac{\partial}{\partial n_\beta} \sum_{\alpha=1}^{N_s} n_\alpha \Psi_\alpha(k) = \Psi_\beta(k) .$$

Substitution of this result for the derivative yields

$$0 = \int_{k_{min}}^{k_{max}} dk \left[\Phi(k)_{theory} - \Phi(k)_{model}\right] \Psi_\beta(k) ,$$

which on substitution from II.4.2 yields

$$\sum_{\alpha=1}^{N_s} n_\alpha \int_{k_{min}}^{k_{max}} dk \; \Psi_\alpha(k) \; \Psi_\beta(k) = \int_{k_{min}}^{k_{max}} dk \; \Phi(k)_{theory} \Psi_\beta(k) .$$

At this point it is convenient to introduce the usual definition of the inner product as

$$(\Psi_\alpha , \Psi_\beta) \equiv \int_{k_{min}}^{k_{max}} dk \; \Psi_\alpha(k) \; \Psi_\beta(k) , \qquad \text{II.4.4}$$

so that the previous result may be compactly written as

$$\sum_{\alpha = 1}^{N_s} n_\alpha \; (\Psi_\alpha , \Psi_\beta) = (\Psi_\beta , \Phi_{theory}) , \quad \text{for } \beta = 1, 2, ..., N_s. \qquad \text{II.4.5}$$

The Uniform Sphere

As a definite example, we consider the case of the homogeneous uniform sphere. The refractive index fluctuation is given by

$$f_\alpha(\vec{r} - \vec{r}_{n_\alpha}) = \begin{cases} A_\alpha, & | \vec{r} - \vec{r}_{n_\alpha} | \leq a_\alpha \\ 0, & \text{otherwise.} \end{cases} \qquad \text{II.4.6}$$

The Fourier transform of the index fluctuation is given by

$$\tilde{f}_{\alpha\vec{k}} = \frac{1}{V} \int_V d^3 \vec{r} \; e^{-i \vec{k} \cdot \vec{r}} \; f_\alpha(\vec{r} - \vec{r}_{n_\alpha})$$

$$= \frac{A_\alpha}{V} \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta \, d\theta \int_0^{a_\alpha} r^2 \, dr \; e^{-i kr \cos\theta}$$

$$= \frac{4\pi}{k^3} \frac{A_\alpha}{V} \left[\sin (ka_\alpha) - ka_\alpha \cos (ka_\alpha) \right] \qquad \text{II.4.7}$$

with the properties that

$$\tilde{f}_{\alpha\vec{k}} = \tilde{f}_{\alpha|\vec{k}|} = \tilde{f}_{\alpha k}, \quad \tilde{f}_{\alpha k} \text{ is real, and } \tilde{f}_{\alpha k} = \tilde{f}_{\alpha(-k)} .$$

Making use of equation II.3.21, we have

$$\Phi(\vec{k}) = \frac{V}{(2\pi)^3} \sum_{\alpha=1}^{N_s} N_\alpha \, \tilde{f}_{\alpha\vec{k}} \, \tilde{f}_{\alpha-\vec{k}} \, .$$

$$= \frac{V}{(2\pi)^3} \sum_{\alpha=1}^{N_s} N_\alpha \left[\frac{4\pi}{k^3} \frac{A_\alpha}{V} \left(\sin(ka_\alpha) - ka_\alpha \cos(ka_\alpha)\right)\right]^2$$

$$= \sum_{\alpha=1}^{N_s} n_\alpha \frac{2 A_\alpha^2}{\pi k^6} \left(\sin(ka_\alpha) - ka_\alpha \cos(ka_\alpha)\right)^2, \qquad \text{II.4.8}$$

where use has been made of the relation

$$\frac{N_\alpha}{V} \rightarrow \text{constant} \equiv n_\alpha \quad \text{as } V \rightarrow \infty \, .$$

We may now make the association with equation II.4.2, identifying that

$$\Psi_\alpha = \frac{2 A_\alpha^2}{\pi k^6} \left(\sin(ka_\alpha) - ka_\alpha \cos(ka_\alpha)\right)^2. \qquad \text{II.4.9}$$

The relations II.4.5 form a matrix equation with the (Ψ_α,Ψ_β) as matrix coefficients relating the unknown vector to the vector (Ψ_β ,Φ_{theory}) . The choice of Φ_{theory} can be dictated by the structure function. We have, for example, $\Phi \propto k^{-11/3}$ for the Kolmogorov spectrum in the inertial subrange.

Within the confines of the uniform sphere model, the choice of magnitude of refractive index fluctuation, A_α, for each size bin is an unspecified parameter. It is still possible, however, to solve the system of equations for the products of the number densities, n_α, with the fluctuation magnitudes, A_α. The Ψ_α can be written as the product of a scalar constant with a function of ka_α. This gives

$$\Psi_\alpha = C_\alpha \, g(ka_\alpha),$$

where

$$C_\alpha = \frac{2 A_\alpha^2 a_\alpha^6}{\pi}$$

and

$$g(ka_\alpha) = \frac{(\sin(ka_\alpha) - ka_\alpha \cos(ka_\alpha))^2}{(ka_\alpha)^6} .$$

With these substitutions, equations II.4.5 can be rewritten as

$$\sum_{\alpha=1}^{N_s} n_\alpha C_\alpha C_\beta(g_\alpha, g_\beta) = C_\beta(g_\beta, \Phi_{theory}), \quad \text{for } \beta = 1, 2, ..., N_s.$$

or

$$\sum_{\alpha=1}^{N_s} [n_\alpha C_\alpha](g_\alpha, g_\beta) = (g_\beta, \Phi_{theory}), \quad \text{for } \beta = 1, 2, ..., N_s. \qquad \text{II.4.10}$$

The system of equations II.4.10 can be solved for the products $n_\alpha C_\alpha$, without prior knowledge of the manner in which the magnitude of the index fluctuation varies with size. We include below a simple subroutine for evaluating the required inner products for the homogeneous uniform sphere model. The routine is written in standard FORTRAN, and so should readily compile on virtually any computing machine, using virtually any compiler.

Numerical Algorithm for Number Densities

```
      program fintk
      dimension a(20),fint(20,20),fint2(20)
c obtain an upper bound on the number of steps to
c allow in the integration process.
      write(6,*)' input integral step limit'
       read(5,*)n
c
c open up the input and output files
      open(2,file='fintk2.inp')
      open(1,file='fintk2.out')
c
c set the chosen number of eddy sizes
      m = 5
c
```

```
c initialize required variables:
c
c   irrational number PI
            pi = 3.141592653
c
c   lower limit of k space integration is 2PI/L, where
c             L is the upper end of inertial subrange eddy
c             size.  L is about 100 meters, =10000 centimeters.
            fk = 2.*pi/10000.
c
c   upper limit of k space integration is 2PI/lo, where
c             lo is the lower end of inertial subrange eddy
c             size.  lo is about 1 centimeter
            uplimk = 2.*pi
c
c   beginning step size dk is one tenth of smallest k
          deltak = .2*pi/10000.
c
c   initialize each integral
            do 700 k=1,m
            do 600 j=1,m
600         fint(k,j) = 0.
700         fint2(k) = 0.
c            note, The fint(k,j) are the inner products
c                  between the g functions.
c                  The fint2(k) are the inner products
c                  between the functions g and phi.
c
c  Read in the chosen eddy sizes
            read(2,*)(A(k),k=1,m)
c
c
c ---------------------------
c
c
c  Perform the integration
            do 1000 i=1,n
            do 900 k = 1,m
            do 800 j = k,m
c   note, the matrix is symmetric, so do upper half only
c
800         fint(k,j) = fint(k,j)+deltak*g(fk*a(k))*g(fk*a(j))
900         fint2(k) = fint2(k)+deltak*g(fk*a(k))*phi(fk)
c
c    increment k by the differential dk
            fk=fk+deltak
c
c    modify step size to get faster run time
            if(deltak .le. 0.01*fk)deltak=1.02*deltak
```

```
c
c    exit loop on reaching upper limit
            if(fk.gt.uplimk)go to 1050
c
c  put a few output statements into the loop to allow
c  monitoring of accumulating integrals.
            if(i.eq.10)write(1,20)fk,((fint(k,j),j=1,m),k=1,m)
     .      (fint2(ii),ii=1,m)
20          format(1x,'k=',f15.6,/,6(5e12.5,/),//)
            if(i.eq.100)write(1,20)fk,((fint(k,j),j=1,m),k=1,m)
     .      (fint2(ii),ii=1,m)
            if(i.eq.300)write(1,20)fk,((fint(k,j),j=1,m),k=1,m)
     .      (fint2(ii),ii=1,m)
            if(i.eq.500)write(1,20)fk,((fint(k,j),j=1,m),k=1,m)
     .      (fint2(ii),ii=1,m)
            if(i.eq.800)write(1,20)fk,((fint(k,j),j=1,m),k=1,m)
     .      (fint2(ii),ii=1,m)
            if(i.eq.900)write(1,20)fk,((fint(k,j),j=1,m),k=1,m)
     .      (fint2(ii),ii=1,m)
            if(i.eq.1000)write(1,20)fk,((fint(k,j),j=1,m),k=1,m)
     .      (fint2(ii),ii=1,m)
            if(i.eq.10000)write(1,20)fk,((fint(k,j),j=1,m),k=1,m)
     .      (fint2(ii),ii=1,m)
1000        continue
c end of integration loop
c ---------------------------------
c
c fill in the lower half of the matrix
1050        do 1200 k=1,m
            do 1100 j=k+1,m
1100        fint(j,k)=fint(k,j)
1200        continue
c
c perform output of final result to output file
            write(1,30)fk
30          format(10x,'k=',f10.3)
            write(1,10)((fint(k,j),j=1,m),k=1,m),(fint2(i),i=1,m)
10          format(1x,5e12.5)
c
c write result to screen
            write(6,10)((fint(k,j),j=1,m),k=1,m),(fint2(i),i=1,m)
c
c that's it...
            stop
            end

            function g(x)
            g = (sin(x) - x*cos(x))**2/x**6
            end
```

```
function phi(x)
phi = x**(-11./3.)
end
```

This routine provides the required matrix elements for evaluating that number density distribution as a function of size parameter within the uniform sphere model, for any particular choice of size bins, which gives the best least squared error to the Kolmogorov spectrum. Any other choice of refractive index model can be used in place of the uniform sphere, by simply replacing the functions g and phi with their corresponding equivalents

Section II.5
Summary of Computation of Number Densities

Our procedure for obtaining the eddy number density distribution corresponding to the atmospheric index of refraction is summarized below.

1. Starting from the structure function of interest, as shown in figure 1,

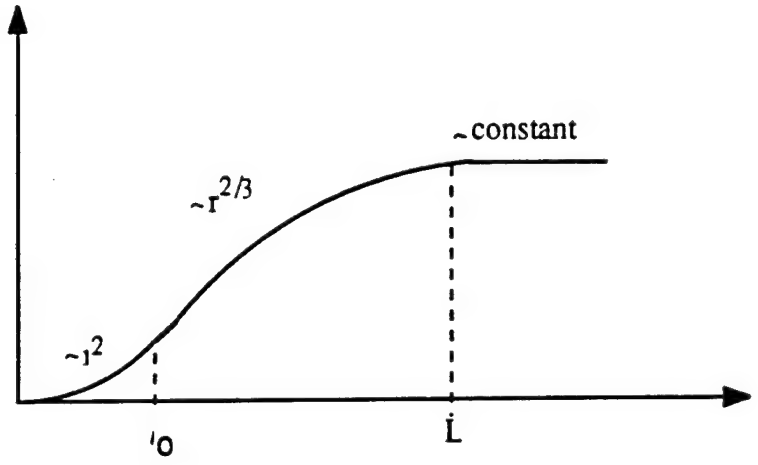

Figure 1. Index of Refraction Structure Function

construct its theoretical spectral density. This step can always be accomplished because the correlation function and the spectral density are Fourier transforms of each other and, by equation II.3.3, the structure function is expressible in terms of correlation functions. Since the structure function consists of three distinct regions, one method is to obtain a spectral density for each region, and impose a continuity requirement.

2. Based on a model of a single eddy, such as the uniform sphere or an exponentially decaying spherical distribution, calculate the model spectral density.

3. Apply the minimum error method to obtain the optimal number density size distribution, i.e. the number of eddies per unit volume of each size, which best matches the model to the theory derived from the structure function.

Part III
Acoustic Scattering from Turbulence

III.1. Introduction

The self consistent field approach is applied to the multiple scattering of acoustic waves from a turbulence distribution consisting of multiple scattering sites of various sizes confined to a turbulence volume. The self consistent field method has previously been successfully applied to the multiple scattering of vector electromagnetic waves. The present work extends the methodology to the multiple scattering of scalar acoustic waves.

In constructing the self consistent field method, it is assumed that the wave scattered by each scatterer is proportional to the wave incident on that scatterer. The wave incident on that single scatterer is assumed to consist of the combination of the wave incident on the system of scatterers and the scattered waves emitted from all other scatterers. Similarly, the waves emitted by all of the other scatters are influenced in part by the emitted wave from the particular single scatterer. In this way, both the wave incident on the single scatterer and the wave emitted by that scatterer include the effects of scattering to all orders. This approach provides a general and systematic procedure for carrying out a perturbation series for the system of scatterers as a whole.

A key factor in the methodology advanced here rests in the computation of a mean field. This involves evaluating averages over the configuration of scatterers before computing the scattered wave, rather than evaluating the scattered wave resulting from particular configurations and later averaging the results. The averaging process for determining the mean field is model dependent. In the succeeding section below, we present the turbulence model employed for these phase I calculations. Subsequent efforts which build from the current work may wish to employ a different or more complex turbulence model, which will require reevaluation of the mean fields.

After describing our turbulence model, we discuss the properties of a single scatterer. This is required as an initial step in the construction of the configuration dependent equations for the scattered field of a system of N scatterers. In particular, these equations are expressed in terms of the exact single scatterer result. Here too, the final results will depend in part on the model selected for the scattering properties of the single site. In the third section of this part of the report, we present an exact result for a particular model of

the single scatterer. Any other model for the single site could also be employed, and approximate, or even empirical solutions could be used when exact solutions are unavailable.

Following the discussion of the single scattering site, we proceed to the consideration of the coupled scattering properties of a collection of N scatterers. This section is very general, and culminates in the presentation of (1) an equation for the total wave in terms of its contributions from the incident wave and the scattered wave from each of the N sites, and (2) a system of self consistent coupled equations satisfied by the set of scattered waves from the N sites. These equations, taken together, fully describe the multiple scattering problem.

Following the development of the equations for the scattering from N sites, we briefly discuss the probabilistic and statistical aspects involved in the characterization of the configuration of the N scatterers. In particular, we discuss the extent to which the probability of finding a particular configuration is or is not factorable into separate probabilities for finding a particular scatterer at a particular given position.

We next proceed to the evaluation of the coherent and the incoherent scattered field. Rather than solving explicitly for the scattered fields and then averaging, the averaging operation is taken first in order to find an approximate equation obeyed by the mean field. The mean field is a much simpler object, depending on the observation point only, and not on the positions or states of the individual scatterers. The coherent and incoherent scattering are then expressed in terms of the mean field.

In preparation for the implementation of the approach in a numerical algorithm, we next present the results of the preceding sections in a plane wave basis. Evaluation of the acoustic signal which would be received at a remote detector is accomplished by means of the far field approximation to the Kirchoff integral. A comparison is made between the current formalism and the "first Born" treatment of Tatarski. It is shown that Tatarski's calculation of the scattered power follows from the more general treatment employed here by keeping lowest order terms only, and making further simplifying assumptions in the averaging process.

In the final sections of the effort, we present the numerical arguments employed in the computational algorithm, as well as provide the text of the code itself. We also discuss the ways in which this code can be incorporated into the shadow zone analysis.

III.2. Turbulence Model

Our model of turbulence consists of a finite concentration of homogeneous, discrete, spherical scatterers of refractive index n and radius a. The scatterers are permitted to be of various sizes, but all scatterers of a given size are identical. To simplify the calculations, all scatterers are assumed to be confined within a rectangular volume, but distributed in a random way, with no correlations between scatterer locations. The independent parameters of the model are: 1) the number density of scatterers of a given size, 2) the refractive index of each scatterer, and 3) the radius of each scatterer. These quantities are fixed by experimental data on the turbulent atmosphere, supplemented by theoretical information such as that provided by the index of refraction structure function.

III.3. The Single Scatterer

The self consistent approach to the multiple scattering problem is a bootstrapping procedure which begins with a solution for the scattering of a wave from a single site. For the current effort, we will work from an exact solution for single site scattering, using the homogeneous sphere model discussed above. An approximate solution using some other model could also be used, provided only that the scattered wave from a particular site is linearly related to the incident wave on that same site.

The linearity assumption is easily represented, using the mathematical formalism of Hilbert spaces. The linearity is preserved whether we use a configuration space representation for the wave, a momentum space representation, or any other representation which lends itself to the selected model. In particular, if we let the incident field be described by the simbol $|\pi_i\rangle$, then the scattered field can be described as a linear function of the incident field:

$$|\pi_s\rangle = \hat{T}\,|\pi_i\rangle\,. \qquad \text{III.3.1}$$

In its most general form, the scattering operator $\hat{T}$ depends on the incident and scattered wave vector, the material properties of the scatterer, and on the geometry of the scatterer,

but not on the fields. The fields $|\pi_s\rangle$ and $|\pi_i\rangle$ are Hilbert space vectors, and the scattering response $\hat{T}$ is an operator in this space. The equation III.3.1 is an abstract and formal representation of the scattering process. In the next paragraph we address the explicit coordinate space representation.

The basic equation for sound propagation in a moving medium can be written (see Tatarski, equation 5.1) in the form

$$\nabla^2 P - \frac{1}{c^2}\left(\frac{\partial}{\partial t} + u_i \frac{\partial}{\partial x_i}\right)^2 P = 0, \qquad \text{III.3.2}$$

where P is the potential of the sound wave, the u_i are the components of the velocity of the medium, and c is the velocity of sound. If we assume a harmonic oscillation for the sound wave, then P can be written as

$$P = \pi e^{-i\omega t}.$$

On inserting this form for P into equation III.3.2, and keeping only terms to first order in the dimensionless fluid velocity u/c, we get the linear result (Tatarski, 5.5)

$$\nabla^2 \pi + k^2 \pi = -2ik \frac{\vec{u}}{c} \cdot \vec{\nabla} \pi + k^2 \frac{t'}{t} \pi, \qquad \text{III.3.3}$$

where $k = \omega/c$ is the wave vector of the sound wave. We note that equation III.3.3 is linear. We further note that the properties of the medium enter into the equation via the fluid velocity $\vec{u}$, and the temperature dependence of the sound velocity c. For simplicity in this initial development of the model, we will drop the velocity dependent term in III.3.3, and consider only temperature fluctuations. Treating the remainder of the right hand side of the equation as a source term, equation III.3.3 may be rewritten as an inhomogeneous integral equation:

$$\pi(\vec{r}) = \pi_0(\vec{r}) - \frac{1}{4\pi} \int dV' \frac{e^{ik|\vec{r} - \vec{r}'|}}{|\vec{r} - \vec{r}'|} k^2 \frac{t'}{t} \pi(\vec{r}') \qquad \text{III.3.4}$$

$$\text{where } \pi_0(\vec{r}) = e^{i \vec{k} \cdot \vec{r}}$$

The second term in III.3.4 describes the scattered wave both close to and far from the scatterer. For a homogeneous spherical scatterer of radius a, equation III.3.3 (or 4) has a well known exact solution. In this case π satisfies a homogeneous equation,

$$\nabla^2\pi + k^2\pi = 0, \qquad \text{III.3.5}$$

outside the scatterer, and a similar equation inside but with a modified wave vector. The solution everywhere is obtained by demanding continuity at the spherical boundary. For scattering by a sphere, we take advantage of the symmetry by choosing functions π that satisfy the homogeneous wave equation in spherical polar coordinates (r, θ, ϕ).

The eigenfunctions $\pi(r, \theta, \phi) = R(r)\,\Theta(\theta)\,\Phi(\phi)$ that are single valued in ϕ (except possibly at points on the boundaries between regions with dissimilar properties) and finite at $\theta = 0$ and $\theta = \pi$ are given by

$$\left.\begin{matrix} \pi_n^{m}{}_{(\text{even})} \\ \pi_n^{m}{}_{(\text{odd})} \end{matrix}\right\} = \left\{\begin{matrix} \cos(m\phi) \\ \sin(m\phi) \end{matrix}\right\} P_n^{m}(\cos\theta)\, Z_n(kr), \qquad \text{III.3.6}$$

where $P_n^{m}(\cos\theta)$ is the associated Legendre function of the first kind of degree n and order m ($n = 0, 1, 2, \ldots, \infty$; $m = 0, 1, 2, \ldots, n$), and $Z_n(kr)$ is any of the four spherical Bessel functions:

$$j_n(\rho) = \sqrt{\frac{\pi}{2\rho}}\; J_{n+1/2}(\rho)$$

$$y_n(\rho) = \sqrt{\frac{\pi}{2\rho}}\; Y_{n+1/2}(\rho)$$

$$h_n^{(1)}(\rho) = j_n(\rho) + iy_n(\rho)$$

$$h_n^{(2)}(\rho) = j_n(\rho) - iy_n(\rho)\,. \qquad \text{III.3.7}$$

Here, $J_{n+1/2}$ and $Y_{n+1/2}$ are the half integral Bessel functions of the first and second kind, j_n and y_n are the spherical Bessel functions (with $\sqrt{\frac{\pi}{2}}$ introduced for convenience), and $h_n^{(1)}$ and $h_n^{(2)}$ are the spherical Bessel functions of the third kind (also known as spherical Hankel functions).

Scattering from a Uniform Sphere

In this section we present a rigorous treatment for scattering from a localized refractive index perturbation which can be modeled as a uniform constant value, different from unity, inside a sphere of radius a, with a refractive index of unity everywhere outside the sphere. The geometry of this configuration is illustrated in figure 2.

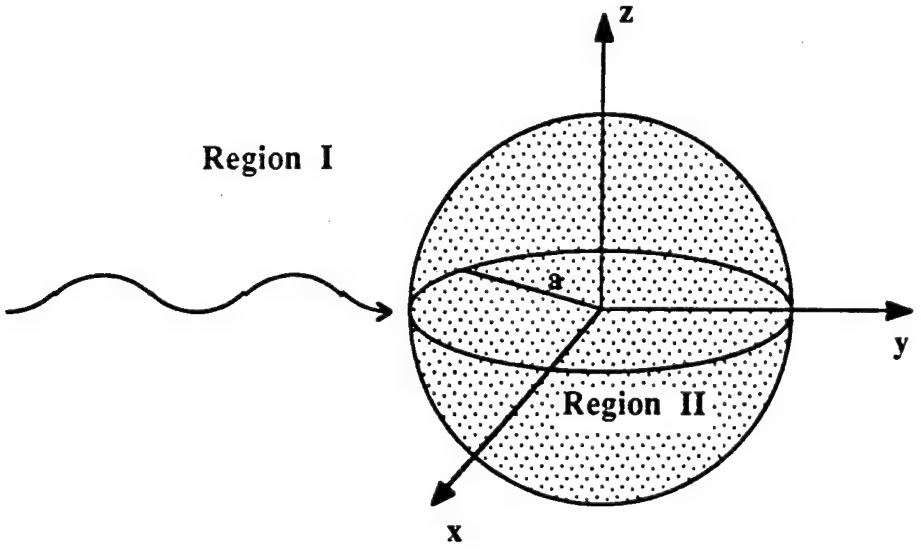

Figure 2. Uniform Sphere Model

Region I is taken to be the region of unperturbed atmosphere outsied the sphere, where the refractive index is unity. Region II is presumed to also be characterized by a constant refractive index, but which differs from unity. In either case, the propagation of sound is characterized by a source free (homogeneous) differential equation:

$$\text{Region I:} \quad (\nabla^2 + k^2)\,\pi_1 = 0$$

$$\text{Region II:} \quad (\nabla^2 + q^2)\,\pi_2 = 0 \qquad \text{III.3.8}$$

We require that the solution should behave as an outgoing spherical wave at infinity, in region I, and that the solution should be finite at the origin, in region II.

At the interface, the two regions are connected by appropriate continuity conditions. In particular, if the pressure on the inside of the interface differs from that on the outside, then the arbitrarily abrupt interface would be subjected to an infinite acceleration. We therefore require that at $r = a$, the pressure must be continuous, i.e., at $r = a$, $p_I = p_{II}$. The pressure continuity requirement is in essence equivalent to conservation of

momentum; if the fluid were to flow with respect to the turbule, we would require conservation at the boundary of the total momentum $p + \rho u^2$ (in a frame fixed to the sphere). The second requirement is that the normal component of velocity should be continuous. This requirement is essentially that of conservation of mass, with the generalization for the flowing fluid $\vec{\nabla} \cdot \rho\vec{u} + \frac{\partial \rho}{\partial t} = 0$, since we do not have sources or sinks at the interface.

An acceptable interior solution can be constructed from those polar coordinate solutions to the homogeneous differential equation which are finite at the origin:

$$\pi_2(r,\theta,\phi) = \sum_{n=0}^{\infty} j_n(qr)\,[a_{n_o} P_n(\cos\theta) + \sum_{m=1}^{n} (a_{nm}\cos m\phi + b_{nm}\sin m\phi) P_n^{\,m}(\cos\theta)]. \qquad \text{III.3.9}$$

Acceptable exterior solutions are constructed from those solutions which behave as an outgoing spherical wave at infinity:

$$\pi_1(r,\theta,\phi) = \sum_{n=0}^{\infty} h_n^{(1)}(kr)[c_{n_o} P_n(\cos\theta) + \sum_{m=1}^{n} (c_{nm}\cos m\phi + d_{nm}\sin m\phi)\, P_n^{\,m}(\cos\theta)]. \qquad \text{III.3.10}$$

In addition to the scattered wave π_1, region I also includes the incident wave:

$$\pi_{inc}(r,\theta,\phi) = e^{ikz} = \sum_{n=0}^{\infty} (2n+1)\, i^n\, j_n(kr)\, P_n(\cos\theta). \qquad \text{III.3.11}$$

The condition of continuity of pressure at the interface can therefore be written as

$$-i\omega\rho'\,\pi_2 = -i\omega\rho(\pi_1 + \pi_{inc}) \quad \ldots \text{ (evaluated at } r = a)$$

or equivalently,

$$\rho' \sum_{n=0}^{\infty} j_n(qa\,[a_{n_o} P_n(\cos\theta) + \sum_{m=1}^{n} (a_{nm} \cos m\phi + b_{nm} \sin m\phi) P_n^m(\cos\theta)]$$

$$= \rho \sum_{n=0}^{\infty} h_n^{(1)}(ka)\,[c_{n_o} P_n(\cos\theta) + \sum_{m=1}^{n} (c_{nm} \cos m\phi + d_{nm} \sin m\phi)\, P_n^m(\cos\theta)]$$

$$+ \rho \sum_{n=0}^{\infty} (2n+1)\, i^n\, j_n(ka)\, P_n(\cos\theta), \qquad \text{III.3.12}$$

from which we find, on equating the coefficients,

$$\rho'\, j_n(qa)\; a_{n_o} = \rho\,[h_n^{(1)}(ka)\, c_{n_o} + (2n+1)\, i^n\, j_n(ka)\,]$$

$$\rho'\, j_n(qa)\, a_{nm} = \rho\; h_n^{(1)}(ka)\; c_{nm} \quad \ldots\ (m \geq 1)$$

$$\rho'\, j_n(qa)\, b_{nm} = \rho\; h_n^{(1)}(ka)\; d_{nm} \quad \ldots\ (m \geq 1)\,. \qquad \text{III.3.13}$$

Similarly, the continuity of the normal component of $\vec{u}$ can be written as

$$-\frac{\partial \pi_2}{\partial r} = -\frac{\partial(\pi_1 + \pi_{inc})}{\partial r} \quad \ldots \text{(evaluated at r=a)},$$

from which we find

$$q\, j_n'(qa)\; a_{n_o} = k\,[h_n'^{(1)}(ka)\, c_{n_o} + (2n+1)\, i^n\, j_n'(ka)\,]$$

$$q\, j_n'(qa)\, a_{nm} = k\, h_n'^{(1)}(ka)\; c_{nm} \quad \ldots\ (m \geq 1)$$

$$q\, j_n'(qa)\, b_{nm} = k\, h_n'^{(1)}(ka)\; d_{nm} \quad \ldots\ (m \geq 1)\,. \qquad \text{III.3.14}$$

For the general k, q, and a, these equations make contrary demands for the relationship between a_{nm} and c_{nm}, which can only be simultaneously satisfied if $a_{nm} = c_{nm} = 0$. The same argument applies to b_{nm} and d_{nm}. We therefore conclude that the $\pi(r,\theta,\phi)$ must be of the form

$$\pi(r,\theta,\phi) = \begin{cases} \sum_{n=0}^{\infty} j_n(qr)\, a_{n_o} P_n(\cos\theta) \;\; ; \; r \le a \\ \sum_{n=0}^{\infty} h_n^{(1)}(kr)\, c_{n_o} P_n(\cos\theta) \;\; ; \; r \ge a, \end{cases} \qquad \text{III.3.15}$$

with a_{n_o} and c_{n_o} given by the solution to the coupled equations

$$\rho' \, j_n(qa) \; a_{n_o} - \rho \, h_n^{(1)}(ka)\, c_{n_o} = \rho \; (2n+1)\, i^n \, j_n(ka)$$

$$q \, j_n'(qa) \; a_{n_o} - k \, h_n'^{(1)}(ka)\, c_{n_o} = k \; (2n+1)\, i^n \, j_n'(ka) \, . \qquad \text{III.3.16}$$

Hilbert Space Formulation

To conclude this section, we return to the fundamental acoustic wave eauation with which we opened, but this time continue the development in our abstract Hilbert space formalism. Using this approach, we will rewrite the equation for sound propagation in the absence of fluid motion, and solve it *exactly* , in a formal manner. Reproducing equation III.3.3 (but neglecting the velocity contribution), we have

$$(\nabla^2 + k^2)\pi = k^2 \frac{t'}{t} \pi \, , \qquad \text{III.3.17}$$

The propagator in the the unperturbed atmosphere is given by

$$(\nabla^2 + k^2)\hat{g}_o = \mathbf{1} \qquad \text{III.3.18}$$

Where **1** indicates the unit matrix or identity operator. On solving this equation for the free propagator $\hat{g}_o$, we get

$$\hat{g}_o = (\nabla^2 + k^2)^{-1} \, . \qquad \text{III.3.19}$$

Equation III.3.17 may now be rewritten in Hilbert Space notation as

$$\hat{g}_o^{-1} \, |\pi\rangle = k^2 \frac{t'}{t} \; |\pi\rangle \, . \qquad \text{III.3.20}$$

The general solution to III.3.20 is given by

$$|\pi\rangle = |\pi_o\rangle + \hat{g}_o \, k^2 \frac{t'}{t} \; |\pi\rangle \, , \qquad \text{III.3.21}$$

where $|\pi_0\rangle$ satisfies a homogeneous equation which can be written in the coordinate space representation as $(\nabla^2 + k^2)\pi_0 = 0$. Equation III.3.21 is an integral equation for $|\pi\rangle$, which can be solved by iteration. The operator $k^2 \frac{t'}{t}$ is responsible for scattering, and is therefore a scattering potential; for simplicity, we rename it to be $\hat{V}'$. Iterating on equation III.3.21, we find

$$|\pi\rangle = |\pi_0\rangle + \hat{g}_0 \hat{V}' |\pi\rangle$$

$$= |\pi_0\rangle + \hat{g}_0 \hat{V}' [\, |\pi_0\rangle + \hat{g}_0 \hat{V}' [\, |\pi_0\rangle + \ldots]]$$

$$= |\pi_0\rangle + \hat{g}_0 \hat{V}' |\pi_0\rangle + (\hat{g}_0 \hat{V}')^2 |\pi_0\rangle + \ldots$$

$$= |\pi_0\rangle + \hat{g}_0 (\hat{V}' + \hat{V}' \hat{g}_0 \hat{V}' + \ldots) |\pi_0\rangle$$

$$= |\pi_0\rangle + \hat{g}_0 \hat{V}' (1 - \hat{g}_0 \hat{V}')^{-1} |\pi_0\rangle$$

$$= |\pi_0\rangle + \hat{g}_0 \hat{S} \, |\pi_0\rangle \qquad \text{III.3.22}$$

where

$$\hat{S} \equiv (1 - \hat{g}_0 \hat{V}')^{-1} \qquad \text{III.3.22}$$

$$= \hat{V}' + \hat{V}' \hat{g}_0 \hat{V}' + \hat{V}' \hat{g}_0 \hat{V}' \hat{g}_0 \hat{V}' + \ldots \quad . \qquad \text{III.3.23}$$

The result III.3.22 is an exact solution to III.3.21, valid to all orders in the scattering potential. If we take the incident field to be given by $|\pi_0\rangle$, then the term $\hat{g}_0 \hat{S} |\pi_0\rangle$ is the exact solution for the scattered wave and is therefore equal to $\hat{T} |\pi_i\rangle$ in equation III.3.1, provided we take the incident field to be $|\pi_i\rangle = |\pi_0\rangle$. In practice, this incident wave is generally chosen to be a plane wave. To summarize these results, we can make the identification

$$\hat{T} = \hat{g}_0 \hat{S} = \hat{g}_0 \hat{V}' (1 - \hat{g}_0 \hat{V}')^{-1} . \qquad \text{III.3.24}$$

Successive orders of approximation to are given by

$$|\pi\rangle = |\pi_0\rangle + |\pi\rangle^{(1)} + |\pi\rangle^{(2)} + \ldots$$

$$|\pi\rangle^{(1)} = \hat{g}_0 \hat{V}' |\pi_0\rangle , \text{ etc.}$$

The contribution $|\pi\rangle^{(1)}$ is the quantity identified as the Born approximation.

Section III.4

The System of N Scatterers

The cornerstone of the development of the scattering formalism for a system of scatterers is the solution to the single site scattering problem given in section III.3:

$$|\pi_s\rangle = \hat{T}\,|\pi_i\rangle\,. \qquad \text{III.4.1}$$

When considering a system of N scatterers, however, the wave incident on a given scattering center is not merely the wave incident on the system, but is rather the combination of that incident wave and all the scattered waves emanating from the other N-1 scattering sites. Let $|p_i\rangle$ represent the acoustic wave incident on the system of scatterers, and let $|\pi_s(\vec{r}, \vec{b}_j)\rangle$ represent the wave scattered by the scattering center located at the site $\vec{b}_j$, where $\vec{b}_j$ is a generalized coordinate comprising both the position $\vec{r}_j$ and size α. The total acoustic wave can be represented by an expression of the form

$$|\pi_T(\vec{r})\rangle = |p_i\rangle + \sum_j |\pi_s(\vec{r}, \vec{b}_j)\rangle\,, \qquad \text{III.4.2}$$

which explicitly states that the total wave is the sum of the incident wave and the waves scattered by each of the N scattering sites.

The total wave is clearly configuration dependent. If we let the notation $\{\vec{b}_j\}$ denote the set of scattering center locations associated with a particular configuration, then this configuration dependence can be made explicit by writing

$$|\pi_T(\vec{r}; \{\vec{b}_j\})\rangle = |p_i\rangle + \sum_j |\pi_s(\vec{r}, \vec{b}_j; \{\vec{b}_j\}')\rangle\,, \qquad \text{III.4.2a}$$

where the notation $\{\vec{b}_j\}' \equiv \{\vec{b}_1, \vec{b}_2, \vec{b}_3, \ldots \vec{b}_{j-1}, \vec{b}_{j+1}, \ldots \vec{b}_N\}$ allows us to specifically extract the dependence on site j, and emphasize the fact that the wave scattered at site j depends on the specific configuration of the other N-1 scatterers.

The wave incident on scatterer j is given by

$$|\pi^j(\vec{r})\rangle = |p_i\rangle + \sum_{\ell \neq j} |\pi_s(\vec{r}, \vec{b}_\ell)\rangle , \qquad \text{III.4.3}$$

$$= |\pi_T(\vec{r})\rangle - |\pi_s(\vec{r}, \vec{b}_j)\rangle . \qquad \text{III.4.3a}$$

Making use of the single scatterer result, III.4.1, the scattered wave from site j can be written as

$$|\pi_s(\vec{r}, \vec{b}_j)\rangle = \hat{T}(\vec{r}; \vec{b}_j)\ |\pi^j(\vec{r})\rangle . \qquad \text{III.4.4}$$

If we substitute the result III.4.4 into the relation III.4.2 for the total wave, we get the result

$$|\pi_T(\vec{r})\rangle = |p_i\rangle + \sum_j \hat{T}(\vec{r}; \vec{b}_j)\ |\pi^j(\vec{r})\rangle . \qquad \text{III.4.5}$$

If we further substitute III.4.4 into III.4.3, we obtain the result that the $|\pi^j(\vec{r})\rangle$ obey a system of linear integral equations

$$|\pi^j(\vec{r})\rangle = |p_i\rangle + \sum_{\ell \neq j} \hat{T}(\vec{r}; \vec{b}_\ell)\ |\pi^\ell(\vec{r})\rangle . \qquad \text{III.4.6}$$

Equations III.4.5 and III.4.6 constitute a set of self-consistent coupled equations which completely determine the multiple scattering problem.

Section III.5
Statistics

Let $p(\{\vec{b}_j\})$ denote the configuration probability density and $\{d^3b_j\}$ the volume element for the N scatterer system. The probability that scatterer 1 will be found in d^3b_1 about $\vec{b}_1$, and that scatterer 2 will be found in d^3b_2 about $\vec{b}_2$, and that scatterer 3 will

be found in d^3b_3 about $\vec{b}_3$, and ..., and that scatterer N will be found in d^3b_N about $\vec{b}_N$, is given by

$$p(\{\vec{b}_j\})\,\{d^3b_j\}\,, \qquad \text{III.5.1}$$

a quantity whose integral is normalized to unity. The probability density for a particular subset n of the N scatterers, such as $\{\vec{b}_1, \vec{b}_2, \vec{b}_3, \ldots \vec{b}_n\,;\, n<N\}$, can be found by integrating the probability density for the N scatterer system over the remaining variables not in the particular subset:

$$p(\{\vec{b}_1, \vec{b}_2, \vec{b}_3, \ldots \vec{b}_n\,;\, n<N\}) = \int d^3b_{n+1} \ldots d^3b_N \; p(\{\vec{b}_j\})\,. \qquad \text{III.5.2}$$

We will denote a completely random distribution, i.e. one which factors completely, by the probability density

$$p_N(\{\vec{b}_j\}) = p_1(\vec{b}_1)\;\; p_2(\vec{b}_2)\;\; p_3(\vec{b}_3) \;\ldots\; p_N(\vec{b}_N) \qquad \text{III.5.3}$$

If the probability density is not completely random, and hence does not factor completely, but rather factors only partially, this can be represented as

$$p(\{\vec{b}_j\}) = p_n(\{\vec{b}_j\})\;\; p_{N-n}(\{\vec{b}_n\}/\{\vec{b}_{N-n}\})\,, \qquad \text{III.5.4}$$

where the last factor denotes the distribution for particles n+1, ..., N, when the values of $\{\vec{b}_n\}$ are known.

The configuration average of the total wave is given by

$$\overline{|\pi_T(\vec{r})\rangle} = \int \{d^3b_j\}\, p(\{\vec{b}_j\})\;\; |\pi_T(\vec{r}\,;\{\vec{b}_j\})\rangle\,. \qquad \text{III.5.5}$$

If n of the scatterers are held fixed, this will be denoted by a subscript:

$$\overline{|\pi_T(\vec{r})\rangle_{\{n\}}} = \int \{d^3b_{N-n}\}\, p_{N-n}(\{\vec{b}_n\}/\{\vec{b}_{N-n}\})\, |\pi_T(\vec{r};\{\vec{b}_j\})\rangle . \qquad \text{III.5.6}$$

The principal use which we will make of expression III.5.6 is to evaluate the configuration average with one of the scatterers, say scatterer k, held fixed, in which particular case expression III.5.6 would be written as

$$\overline{|\pi_T(\vec{r})\rangle_k} = \int d^3b_1 d^3b_2 \ldots d^3b_{k-1} d^3b_{k+1} \ldots d^3b_N\, p_{N-1}(\vec{b}_k/\{\vec{b}_k'\})\, |\pi_T(\vec{r};\{\vec{b}_j\})\rangle \qquad \text{III.5.7}$$

Note that probability distributions may be converted to density distributions by multiplying by the appropriate power of the number of scatterers, for example:

$$n(\vec{b}_1) = N\, p_1(\vec{b}_1); \quad n(\vec{b}_1,\vec{b}_2) = N^2\, p_2(\vec{b}_1,\vec{b}_2) . \qquad \text{III.5.8}$$

Section III.6
The Coherent and Incoherent Waves

It is generally not possible to obtain the total wave for a given configuration of scatterers, nor is it necessary, since we are interested only in the first and second moments of the N scatterer probability distribution. In this section we will use the statistical relations of the preceding section to perform the required averages, and obtain averaged expressions for the coherent and incoherent contributions to the scattered wave.

Coherent Scattering

The average over III.4.5 may be performed using relations III.5.5 and III.5.6, along with relation III.5.4, with n = 1.

$$\overline{|\pi_T\rangle} = |p_i\rangle + \sum_j \int d^3b_j\, p_1(\vec{b}_j)\, \hat{T}(\vec{b}_j)\, \overline{|\pi^j\rangle_j} , \qquad \text{III.6.1}$$

where the object

$$\overline{|\pi^j\rangle_j} \equiv \overline{|\pi^j(\vec{r},\vec{b}_j)\rangle_j}$$

$$= \int d^3b_1\, d^3b_2 \ldots d^3b_{j-1}\, d^3b_{j+1} \ldots d^3b_N \; p_{N-1}(\vec{b}_j / \{\vec{b}_j\}') \; |\pi^j(\vec{r};\{\vec{b}_j\})\rangle . \qquad \text{III.6.2}$$

defines the effective field, and represents the field incident on the particular scatterer j whose position is known, averaged over the distribution of all other scatterers. Note that the $\vec{r}$ dependence is still implied, but has been dropped from the explicit notation in relation III.6.1.

The effective field differs from the total field by the field emitted by one scatterer. If the number of scatterers, N, is large, then the effective field is approximately equal to the coherent field:

$$\overline{|\pi^j\rangle_j} \equiv \overline{|\pi_T(\vec{r})\rangle} \; . \qquad \text{III.6.3}$$

This last statement is true for each of the N scatterers, so that when this approximation is inserted into relation III.6.1, the summation over j is simply replaced by a multiplicative factor N. In particular we have

$$\overline{|\pi_T\rangle} = |p_i\rangle + \overline{\hat{T}} \; \overline{|\pi_T\rangle} \; , \qquad \text{III.6.4}$$

where

$$\overline{\hat{T}} = N \int d^3b_j \;\; p_1(\vec{b}_j) \;\; \hat{T}(\vec{b}_j), \quad \text{and} \quad N\, p_1(\vec{b}_j) = n(\vec{b}_j). \qquad \text{III.6.5}$$

This approximation replaces solving the system of N coupled linear equations for the total field (relations III.4.5 and III.4.6) by a single integral equation for the coherent field. Iterating III.6.4 we get

$$\overline{|\pi_T\rangle} = (1 + \overline{\hat{T}} + (\overline{\hat{T}})^2 + \ldots)\,|p_i\rangle$$

$$= (1 - \overline{\hat{T}})^{-1}\,|p_i\rangle\,. \qquad \text{III.6.6}$$

Incoherent Scattering

The density and directionality of incoherent radiation is given by the $\langle \vec{r} \,|\ldots|\, \vec{r_0} \rangle$ matrix elements of the quantity $|\pi_T\rangle\langle\pi_T|$. We now proceed to obtain an equation for this quantity, starting from the basic relation III.4.5 and its dual. In the following, we will drop the explicit $\vec{r}$ and $\{\vec{b}_j\}$ dependences from the explicit notation, leaving those arguments as merely implied. Specifically, we get

$$|\pi_T\rangle = |p_i\rangle + \sum_j \hat{T}_j\,|\pi^j\rangle \qquad \text{III.6.7}$$

and

$$\langle\pi_T| = \langle p_i| + \sum_j \langle\pi^j|\,T_j^{\dagger}\,. \qquad \text{III.6.8}$$

On taking the difference between the average of the product and the product of the averages of equations III.6.7 and III.6.8, we get

$$\overline{|\pi_T\rangle\langle\pi_T|} = \overline{|\pi_T\rangle}\;\overline{\langle\pi_T|} + \overline{\sum_{jk} \hat{T}_j\,|\pi^j\rangle\langle\pi^k|\hat{T}_k^{\dagger}} - \overline{\hat{T}_j\,|\pi^j\rangle}\;\overline{\langle\pi^k|\hat{T}_k^{\dagger}} \qquad \text{III.6.9}$$

The averages of the diagonal and off diagonal terms of the second quantity on the right hand side of relation III.6.9 are:

$$\overline{\hat{T}_j\,|\pi^j\rangle\langle\pi^j|\hat{T}_j^{\dagger}} = \int d^3b_j\; p_1(\vec{b}_j)\;\hat{T}(\vec{b}_j)\;(\,\overline{|\pi^j\rangle\langle\pi^j|}\,)_j\;\hat{T}^{\dagger}(\vec{b}_j) \qquad \text{III.6.10}$$

and

$$\overline{\hat{T}(\vec{b}_j)\,|\pi^j\rangle\langle\pi^k|\hat{T}^{\dagger}(\vec{b}_k)}$$

$$= \int d^3b_j\, d^3b_k p_2(\vec{b}_j,\vec{b}_k)\hat{T}(\vec{b}_j)\,(\,\overline{|\pi^j\rangle\langle\pi^k|}\,)_{jk}\hat{T}^{\dagger}(\vec{b}_k). \qquad \text{III.6.11}$$

Relations III.6.9 through III.6.11 are exact. To simplify the problem we make similar approximations to those made in the coherent case:

$$\overline{(\,|\pi^j\rangle\,)}_j \equiv \overline{|\pi_T\rangle}\ ,$$

$$\overline{(\,|\pi^j\rangle\langle\pi^j|\,)}_j \equiv \overline{|\pi_T\rangle\langle\pi_T|}\ ,$$

and

$$\overline{|\pi^j\rangle\langle\pi^k|}_{jk} \equiv \overline{|\pi_T\rangle\langle\pi_T|} \qquad \text{III.6.12}$$

At this point, it is convenient to introduce three super-operators, R, M, and Q . These super-operoperators are defined for and operate on any given quantity X which is independent of the configuration of the scatterers and their states. The operators are defined by:

$$R\ X \equiv \overline{\hat{T}}\ X\ \overline{\hat{T}}^{\dagger}$$

$$M\ X \equiv N \int d^3b_j\ \ p_1(\vec{b}_j)\ \ \hat{T}(\vec{b}_j)\ X\ \overline{\hat{T}}^{\dagger}(\vec{b}_j)$$

and

$$Q\ X \equiv N(N\text{-}1)\int d^3b_j\, d^3b_k\, p_2(\vec{b}_j,\vec{b}_k)\hat{T}(\vec{b}_j)\ X\ (\hat{T}^{\dagger}(\vec{b}_k). \qquad \text{III.6.13}$$

Using relations III.6.12 and the super-operators III.6.13, it is possible to rewrite equation III.6.9 as

$$\overline{|\pi_T\rangle\langle\pi_T|} = \overline{|\pi_T\rangle}\ \overline{\langle\pi_T|} + (M+Q)\ \overline{|\pi_T\rangle\langle\pi_T|} - R\ \overline{|\pi_T\rangle}\ \overline{\langle\pi_T|}. \qquad \text{III.6.14}$$

If we add to both sides the quantity $R\ (\overline{|\pi_T\rangle}\ \overline{\langle\pi_T|} - \overline{|\pi_T\rangle\langle\pi_T|}\,)$, we get

$$(1-R)\,(\,\overline{|\pi_T\rangle\langle\pi_T|} - \overline{|\pi_T\rangle}\ \overline{\langle\pi_T|}\,) = (M+Q-R)\ \overline{|\pi_T\rangle\langle\pi_T|}\,, \qquad \text{III.6.15}$$

which is readily written as an inhomogeneous integral equation:

$$\overline{|\pi_T\rangle\langle\pi_T|} = \overline{|\pi_T\rangle}\ \overline{\langle\pi_T|} + L\ \overline{|\pi_T\rangle\langle\pi_T|}\ , \qquad \text{III.6.16}$$

where

$$L = (1 - R)^{-1}\ (M + Q - R)\ . \qquad \text{III.6.17}$$

Upon iterative expansion of III.6.16, we obtain the following power series for the incoherent radiation:

$$\overline{|\pi_T\rangle\langle\pi_T|} = (1 + L + L^2 + L^3 + \ldots)\ \overline{|\pi_T\rangle}\ \overline{\langle\pi_T|}$$

$$= (1 - L)^{-1}\ \overline{|\pi_T\rangle}\ \overline{\langle\pi_T|}\ . \qquad \text{III.6.18}$$

From III.6.18, it is evident that the coherent radiation term $\overline{|\pi_T\rangle}\ \overline{\langle\pi_T|}$ acts as a source term for the incoherent radiation $\overline{|\pi_T\rangle\langle\pi_T|}$. The composite superoperator L generates successive powers of incoherent scattering. On neglecting L , or equivalently on considering only the L^0 term in III.6.18, the radiation is all coherent and we have

$$\overline{|\pi_T\rangle\langle\pi_T|} \cong \overline{|\pi_T\rangle}\ \overline{\langle\pi_T|}\ . \qquad \text{III.6.19}$$

The operator$L$ generates only incoherent scattering, with successive powers representing radiation that has been incoherently scattered once, twice, and so on. The M term represents the purely incoherent addition of the the intensities of individual scatterers, while the term $Q - R$ represents scattering due to the correlation $p_2(\vec{b_1}, \vec{b_2}) - p_1(\vec{b_1})p_1(\vec{b_2})$ between scatterer locations.

Section III.7

Coherent and Incoherent Scattering in a Plane Wave Representation

In this section we recast the principal results III.6.6, III.6.18, and III.6.19 in a plane wave representation. The motivation for employing plane waves for the system of N scatterers is that the configuration dependence appears, in this representation, as a simple multiplicative phase factor. This simplifies the averaging procedure.

The Plane Wave Basis

We will consider a cubical box $V = L^3$, which encloses the distribution of scatterers. Plane waves which satisfy periodic boundary conditions at the walls of the box will have wave vectors $\vec{k}_\sigma = \frac{2\pi\vec{\sigma}}{L}$, with $\vec{\sigma} = \sigma_1\hat{x} + \sigma_2\hat{y} + \sigma_3\hat{z}$; $\sigma_1, \sigma_2, \sigma_3$ all integers. We can denote a complete set of orthonormal state vectors indexed by the wave vectors, using the Dirac bracket notation, as $|\vec{k}_\sigma\rangle$. This set of plane wave states forms a complete orthonormal set, with the properties of completeness and orthnormality given by

$$\sum_\sigma |\vec{k}_\sigma\rangle\langle\vec{k}_\sigma| = \mathbb{1} \qquad \text{III.7.1}$$

and

$$\langle\vec{k}_\sigma|\vec{k}_{\sigma'}\rangle = \delta_{\sigma\sigma'}. \qquad \text{III.7.2}$$

If we introduce a momentum operator $\vec{p} = -i\vec{\nabla}$, then this operator takes on wave vector eigenvalues so that (neglecting the size parameter α for simplicity)

$$e^{i\vec{p}\cdot\vec{r}_j}|\vec{k}_{\sigma'}\rangle = e^{i\vec{k}_\sigma\cdot\vec{r}_j}|\vec{k}_{\sigma'}\rangle, \qquad \text{III.7.3}$$

where $\vec{r}_j$ is any of the scatterer locations. We use this property to compute the matrix elements of $\hat{T}(\vec{r}_j)$ between plane wave states. In particular, we have

$$\hat{T}(\vec{r}_j) = e^{-i\vec{p}\cdot\vec{r}_j}\,\hat{T}(\vec{0})\,e^{i\vec{p}\cdot\vec{r}_j}, \qquad \text{III.7.4}$$

so that the matrix elements of the transition operator are

$$\langle \vec{k}_{\sigma'} | \hat{T}(\vec{r}_j) | \vec{k}_{\sigma} \rangle = \langle \vec{k}_{\sigma'} | e^{-i \vec{p} \cdot \vec{r}_j} \hat{T}(\vec{0}) e^{i \vec{p} \cdot \vec{r}_j} | \vec{k}_{\sigma} \rangle$$

$$= e^{i (\vec{k}_{\sigma} - \vec{k}_{\sigma'}) \cdot \vec{r}_j} \langle \vec{k}_{\sigma'} | \hat{T}(\vec{0}) | \vec{k}_{\sigma} \rangle . \qquad \text{III.7.5}$$

The meaning of III.7.5 is that the matrix element of the scattering amplitude for a single scatterer between plane wave states $| \vec{k}_{\sigma} \rangle$ and $| \vec{k}_{\sigma'} \rangle$ for a scatterer at $\vec{r}_j$ differs from that of a scatterer at the origin only by the phase factor $e^{i (\vec{k}_{\sigma} - \vec{k}_{\sigma'}) \cdot \vec{r}_j}$. An important point here is that the random variable $\vec{r}_j$ appears only in this phase factor.

Coherent and Incoherent Scattering

We will begin the development of the formalism in the plane wave basis by computing the expression III.6.6 for the mean total coherent wave. This is accomplished by computing successive terms in the series expansion of $\overline{|\pi_T\rangle}$, and performing the infinite sum. We will assume that the scatterers are distributed completely randomly throughout the volume $V = L^3$ which encloses the scatterers, so that the number density $n(\vec{r}_j)$ is given by $n(\vec{r}_j) = N\, p(\vec{r}_j) = N/V$.

Specifically. equation III.6.6 says:

$$\overline{|\pi_T\rangle} = (1 + \overline{\hat{T}} + (\overline{\hat{T}})^2 + \ldots) |p_i\rangle$$

$$= (1 - \overline{\hat{T}})^{-1} |p_i\rangle .$$

In order to convert this to a plane wave basis, we can invoke the definition of the mean value of the single scatterer transition operator to obtain

$$\overline{\hat{T}} = N \times \frac{1}{V} \int_V d^3 r_j \, \hat{T}(\vec{r}_j)$$

$$= \frac{N}{V} \int_V d^3 r_j \sum_{\sigma \sigma'} | \vec{k}_{\sigma'} \rangle \langle \vec{k}_{\sigma'} | \hat{T}(\vec{r}_j) | \vec{k}_{\sigma} \rangle \langle \vec{k}_{\sigma} |$$

$$= \frac{N}{V} \int_V d^3 r_j \sum_{\sigma\sigma'} | \vec{k}_{\sigma'} \rangle \, e^{i(\vec{k}_\sigma - \vec{k}_{\sigma'}) \cdot \vec{r}_j} \langle \vec{k}_{\sigma'} | \hat{T}(\vec{0}) | \vec{k}_\sigma \rangle \langle \vec{k}_\sigma |$$

$$= N \sum_\sigma \langle \vec{k}_\sigma | \hat{T}(\vec{0}) | \vec{k}_\sigma \rangle | \vec{k}_\sigma \rangle \langle \vec{k}_\sigma | \qquad \text{III.7.6}$$

$$\equiv N \sum_\sigma T_\sigma | \sigma \rangle \langle \sigma | \, . \qquad \text{III.7.7}$$

Making use of this result, we can compute the second power of the average transition operator:

$$(\overline{\hat{T}})^2 = \overline{\hat{T}} \cdot \overline{\hat{T}} = (N \sum_\sigma T_\sigma | \sigma \rangle \langle \sigma |) \times (N \sum_{\sigma'} T_{\sigma'} | \sigma' \rangle \langle \sigma' |)$$

$$= N^2 \sum_{\sigma\sigma'} T_\sigma T_{\sigma'} | \sigma \rangle \langle \sigma | \sigma' \rangle \langle \sigma' |$$

$$= N^2 \sum_{\sigma\sigma'} T_\sigma T_{\sigma'} | \sigma \rangle \, \delta_{\sigma\sigma'} \, \langle \sigma' |$$

$$= \sum_\sigma (NT_\sigma)^2 | \sigma \rangle \langle \sigma |$$

The same argument can be made to show that the general term is

$$(\overline{\hat{T}})^k = \sum_\sigma (NT_\sigma)^k | \sigma \rangle \langle \sigma | \, ,$$

so that the full series $(1 - \overline{\hat{T}})^{-1}$ is given by

$$(1 - \overline{\hat{T}})^{-1} = \sum_\sigma [1 + NT_\sigma + (NT_\sigma)^2 + \ldots] | \sigma \rangle \langle \sigma |$$

$$= \sum_\sigma \frac{1}{1 - NT_\sigma} | \sigma \rangle \langle \sigma | \, . \qquad \text{III.7.8}$$

On substituting this into III.6.6, we obtain

$$\overline{|\pi_T\rangle} = (1 - \overline{\hat{T}})^{-1} |p_i\rangle = \sum_\sigma \frac{1}{1 - NT_\sigma} \langle \sigma | p_i \rangle | \sigma \rangle \qquad \text{III.7.9}$$

The coherent scattering, III.6.19, is then given by

$$\overline{|\pi_T\rangle\langle\pi_T|} \equiv \overline{|\pi_T\rangle}\ \overline{\langle\pi_T|} = \sum_{\sigma\sigma'} \frac{\langle\sigma|p_i\rangle}{1 - NT_\sigma} \frac{\langle\sigma'|p_i\rangle^*}{1 - NT^*_{\sigma'}} |\sigma\rangle\langle\sigma'|. \quad \text{III.7.10}$$

We will also evaluate the incoherent scattering contribution to first order in the generating operator L. From III.6.18 in the limit of small L, we have to first order

$$\overline{|\pi_T\rangle\langle\pi_T|} \equiv (1 + L)\ \overline{|\pi_T\rangle}\ \overline{\langle\pi_T|} \quad \text{III.7.11}$$

We show in appendix A that the first order incoherent term is given by

$$\overline{|\pi_T\rangle\langle\pi_T|} - \overline{|\pi_T\rangle}\ \overline{\langle\pi_T|} = L\ \overline{|\pi_T\rangle}\ \overline{\langle\pi_T|}$$

$$= N \sum_{\sigma\sigma'\sigma''} \frac{T_{\sigma\sigma''} T^*_{\sigma'\sigma''+\sigma'-\sigma}}{1 - NT_\sigma NT^*_{\sigma'}} \frac{\langle\sigma''|p_i\rangle}{1 - NT_{\sigma''}} \frac{\langle\sigma''+\sigma'-\sigma|p_i\rangle}{1 - NT^*_{\sigma''+\sigma'-\sigma}} |\sigma\rangle\langle\sigma'|. \quad \text{III.7.12}$$

For evaluating the scattering cross section, we will be interested in computing the matrix elements $\langle\vec{r}|\ldots|\vec{r}'\rangle$ of the quantities III.7.10 and III.7.12. This will be addressed in section III.8, below.

To conclude the current section, we discuss the calculation of the matrix elements T_σ and $T_{\sigma\sigma'}$. First we compute T_σ. The definition of T_σ is

$$T_\sigma \equiv \langle \vec{k}_\sigma |\ \hat{T}(\vec{0})\ |\vec{k}_\sigma\rangle ,$$

which can be written in configuration space as

$$T_\sigma = \int d^3r\ \langle \vec{k}_\sigma|\vec{r}\rangle\langle\vec{r}|\hat{T}(\vec{0})\ |\vec{k}_\sigma\rangle , \quad \text{III.7.13}$$

where

$$\langle \vec{r}|\vec{k}_\sigma\rangle = \frac{1}{\sqrt{V}}\ e^{i\vec{k}_\sigma\cdot\vec{r}} = \langle \vec{k}_\sigma|\vec{r}\rangle^* . \quad \text{III.7.14}$$

The quantity $\langle\vec{r}|\hat{T}(\vec{0})\ |\vec{k}_\sigma\rangle$ is the effect of the single scatterer on a simple plane wave, as expressed in the configuration space representation. This is therefore the same

quantity as was evaluated as the exterior solution in equations III.3.15 for the single uniform sphere scatterer, and is given as

$$\langle \vec{r} \mid \hat{T}(\vec{0}) \mid \vec{k}_\sigma \rangle = \sum_{n=0}^{\infty} c_{n_o} h_n^{(1)}(k_\sigma r) P_n(\cos\theta) . \qquad \text{III.7.15}$$

On substituting III.7.14 and III.7.15 into III.7.13, we get

$$T_\sigma = \frac{1}{\sqrt{V}} \int d^3r \; e^{-i \vec{k}_\sigma \cdot \vec{r}} \sum_{n=0}^{\infty} c_{n_o} h_n^{(1)}(k_\sigma r) P_n(\cos\theta)$$

$$= \frac{1}{\sqrt{V}} \sum_{n=0}^{\infty} c_{n_o} \int d^3r \; P_n(\cos\theta) \; e^{-i k_\sigma r \cos\theta} \; h_n^{(1)}(k_\sigma r) , \qquad \text{III.7.16}$$

where an arbitrary choice of the configuration space coordinate system orientation has been made, placing the z axis parallel to $\vec{k}_\sigma$, so that $\vec{k}_\sigma \cdot \vec{r} = k_\sigma r \cos\theta$, which simplifies the computations.

Next, we evaluate the off diagonal terms, $T_{\sigma\sigma'}$. Once again we start from the definition

$$T_{\sigma\sigma'} \equiv \langle \vec{k}_\sigma \mid \hat{T}(\vec{0}) \mid \vec{k}_{\sigma'} \rangle$$

$$= \int d^3r \; \langle \vec{k}_\sigma \mid \vec{r} \rangle \langle \vec{r} \mid \hat{T}(\vec{0}) \mid \vec{k}_{\sigma'} \rangle . \qquad \text{III.7.17}$$

This time we allign the z axis along $\vec{k}_{\sigma'}$ so that

$$\langle \vec{r} \mid \hat{T}(\vec{0}) \mid \vec{k}_{\sigma'} \rangle = \sum_{n=0}^{\infty} c_{n_o} h_n^{(1)}(k_{\sigma'} r) P_n(\cos\theta) . \qquad \text{III.7.18}$$

It is convenient to write $\vec{r}$ and $\vec{k}_\sigma$ in polar coordinates, with reference to figure 3. In particular, we have

$$\vec{k}_\sigma \; \underset{(\text{cartesian})}{=} \; (k_{\sigma_x}, k_{\sigma_y}, k_{\sigma_z}) \; \underset{(\text{polar})}{=} \; (k_\sigma, \alpha, \beta)$$

and

$$\vec{r} \underset{(\text{cartesian})}{=} (x, y, z) \underset{(\text{polar})}{=} (r, \theta, \phi) .$$

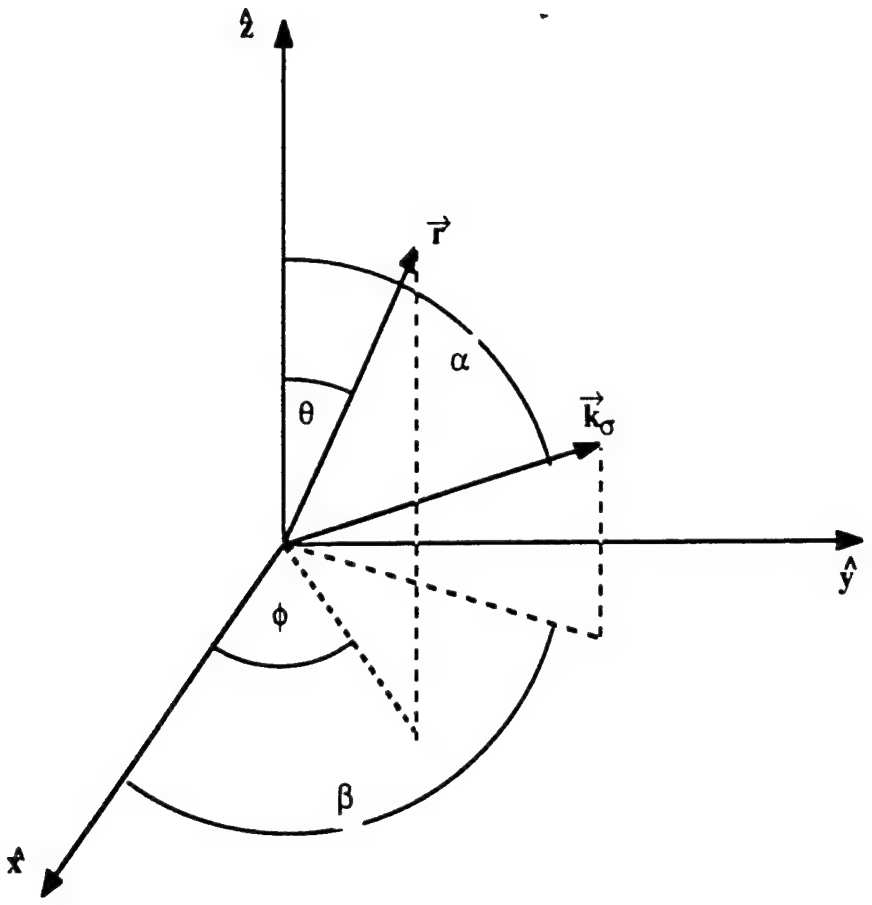

Figure 3. Coordinate System

On explicitly writing the cartesian components in terms of the radial components and the polar and azimuthal angles, we obtain:

$$k_{\sigma_x} = k_\sigma \sin\alpha \cos\beta$$
$$k_{\sigma_y} = k_\sigma \sin\alpha \sin\beta$$
$$k_{\sigma_z} = k_\sigma \cos\alpha$$

and

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

so that

$$\vec{k}_\sigma \cdot \vec{r} = k_\sigma r\,[\sin\alpha\sin\theta\cos(\phi-\beta) + \cos\alpha\cos\theta]$$

$$\equiv k_\sigma r\cos\gamma, \text{ where } \gamma \text{ is the angle between } \vec{r} \text{ and } \vec{k}_\sigma.$$

Therefore we finally obtain

$$T_{\sigma\sigma'} = \frac{1}{\sqrt{V}} \sum_{n=0}^{\infty} c_{n_o} \int d^3r\, P_n(\cos\theta)\, e^{-i k_\sigma r\cos\gamma}\, h_n^{(1)}(k_{\sigma'} r)\,. \qquad \text{III.7.19}$$

Section III.8
Far Field Solution and Scattering Cross Section

Far Field Solution

The approach used in this section for characterizing the far field solution makes use of Kirchoff's method for expressing a scalar field inside a closed volume in terms of the value of the field and its normal derivative on the boundary surfaces of the volume. In the current instance, we will consider the volume of interest to be the portion of space outside of the turbulence region, and extending to infinity. For the purposes of visualization, we might consider the volume of interest to be as shown in figure 4, taken to the limit in which the outer box extends to infinity. We will be interested in describing the scalar acoustic field and its normal derivative at the surface S_1 of the turbulence volume, and at the surface at infinity, S_2.

For the given acoustic potential $P(\vec{r},t) = e^{-i\omega t}\pi_s(\vec{r})$, with $\pi_s(\vec{r})$ satisfying $(\nabla^2 + k^2)\,\pi_s(\vec{r}) = 0$, the free space Green's function will satisfy

$$(\nabla^2 + k^2)\, G(\vec{r},\vec{r}') = -\delta(\vec{r}-\vec{r}')\,. \qquad \text{III.8.1}$$

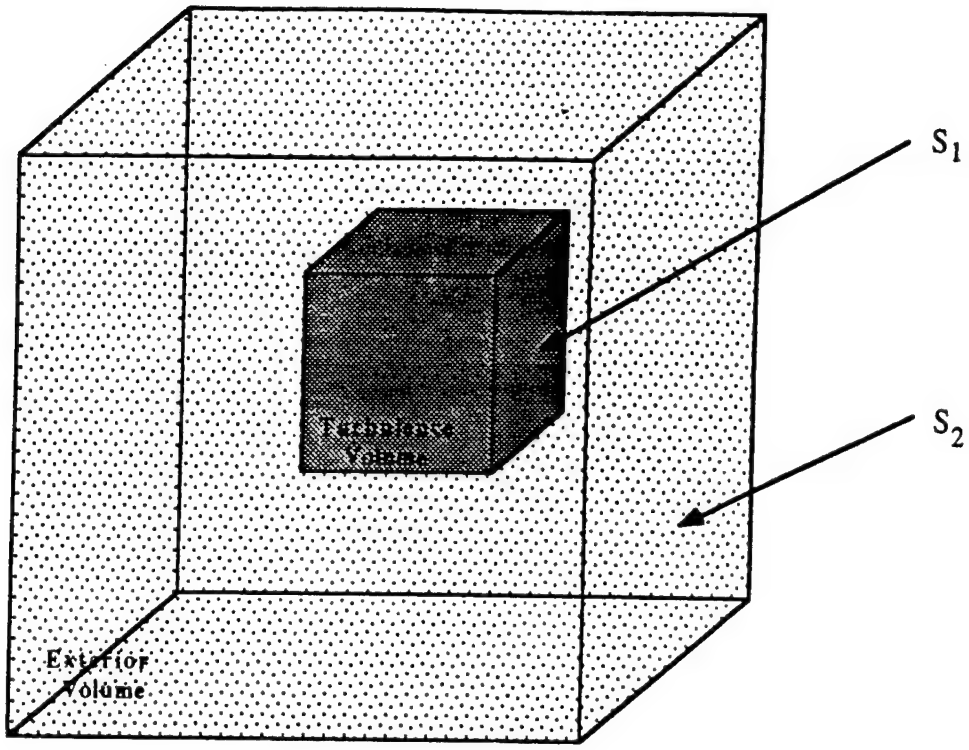

Figure 4. Far Field Volume

Green's theorem allows the characterization of the field $\pi_s(\vec{r})$ in terms of a surface integral

$$\pi_s(\vec{r}) = \oint_S [\, \pi_s(\vec{r}')\, \hat{n}' \cdot \nabla' G(\vec{r},\vec{r}') - G(\vec{r},\vec{r}')\, \hat{n}' \cdot \nabla' \pi_s(\vec{r}')\,]\, da', \qquad \text{III.8.2}$$

where $\hat{n}'$ is the inwardly directed normal to the surface S .

The free space Green's function is given by

$$G(\vec{r},\vec{r}') = \frac{e^{ikR}}{4\pi R}, \qquad \text{III.8.3}$$

where $\vec{R} = \vec{r} - \vec{r}'$, and $R = |\vec{R}|$.

On inserting this expression into III.8.2, we find

$$\pi_s(\vec{r}) = -\frac{1}{4\pi} \oint_S \frac{e^{ikR}}{R} \hat{n}' \cdot [\nabla' \pi_s(\vec{r}') + ik(1+\frac{i}{kR})\frac{\vec{R}}{R} \pi_s(\vec{r}')]da' , \qquad \text{III.8.4}$$

where $S = S_1 + S_2$, as is shown in figure 4, V is the volume enclosed by S_1 and S_2, and the normal $\hat{n}'$ is directed into V from S_1 and S_2.

In the neighborhood of the surface S_2 at infinity, the scattered wave will behave as an outgoing wave, with the implication that

$$\pi_s(\vec{r}) \sim \frac{e^{ikr}}{4\pi r} f(\theta,\phi)$$

and

$$\frac{1}{\pi_s(\vec{r})} \frac{\partial \pi_s(\vec{r})}{\partial r} \sim (ik - \frac{1}{r}) ,$$

so that the integral III.8.4 vanishes on S_2 at least as fast as $1/r$. We may therefore reduce III.8.4 to an integral over S_1 only:

$$\pi_s(\vec{r}) = -\frac{1}{4\pi} \oint_{S_1} \frac{e^{ikR}}{R} \hat{n}' \cdot [\nabla' \pi_s(\vec{r}') + ik(1+\frac{i}{kR})\frac{\vec{R}}{R} \pi_s(\vec{r}')]da' . \qquad \text{III.8.5}$$

The contributions to this integral are derived from the results for π_T of the previous section, coupled with the fact that $\pi_s = \pi_T - \pi_i$.

If the observation point is far from the scattering region, then

$$G(\vec{r},\vec{r}') \rightarrow \frac{e^{ikr}}{4\pi r} e^{-ikr'} \qquad \text{III.8.6}$$

and

$$\pi_s(\vec{r}) \rightarrow \frac{e^{ikr}}{r} F(\vec{k}, \vec{k}_{inc}) \qquad \text{III.8.7}$$

In addition, we have

$$\pi_s(\vec{r}) = \frac{e^{ikr}}{r} \frac{1}{4\pi} \oint_{S_1} e^{-i\vec{k}\cdot\vec{r}'} [-i\vec{k}\cdot\hat{n}'\,\pi_s(\vec{r}') - \hat{n}'\cdot\nabla'\,\pi_s(\vec{r}')]da', \qquad \text{III.8.8}$$

from which we can identify that

$$F(\vec{k}, \vec{k}_{inc}) = \frac{1}{4\pi} \oint_{S_1} e^{-i\vec{k}\cdot\vec{r}'} [-i\vec{k}\cdot\hat{n}'\,\pi_s(\vec{r}') - \hat{n}'\cdot\nabla'\,\pi_s(\vec{r}')]da'$$

$$= \frac{1}{4\pi} \oint_{S_1} e^{-i\vec{k}\cdot\vec{r}'} \hat{n}'\cdot [-i\vec{k}\,\pi_s(\vec{r}') - \nabla'\,\pi_s(\vec{r}')]da'. \qquad \text{III.8.9}$$

Differential Scattering Cross Section

The differential scattering cross section is defined by

$$\frac{d\sigma(\vec{k}, \vec{k}_{inc})}{d\Omega} \equiv \frac{\text{power scattered in direction } \vec{k}}{\text{solid angle} \times \text{incident flux in direction } \vec{k}_{inc}}, \qquad \text{III.8.10}$$

where we are interested in time averaged, rather than instantaneous values of flux. The power scattered in direction $\vec{k}$ is the scalar product of the directed flux $\vec{S}$ with the oriented area $\vec{A}$:

$$P = \vec{S}\cdot\vec{A} \qquad \text{III.8.11}$$

The time averaged directed flux for harmonic fields is given by (see Tatarski, ch 5)

$$\overline{\langle \vec{S} \rangle} = \frac{\omega\rho}{2} \operatorname{Im}\overline{(\pi\nabla\pi)}, \qquad \text{III.8.12}$$

where ρ is the density of air. We show the selected coordinate system geometry for the scattering process in figure 5.

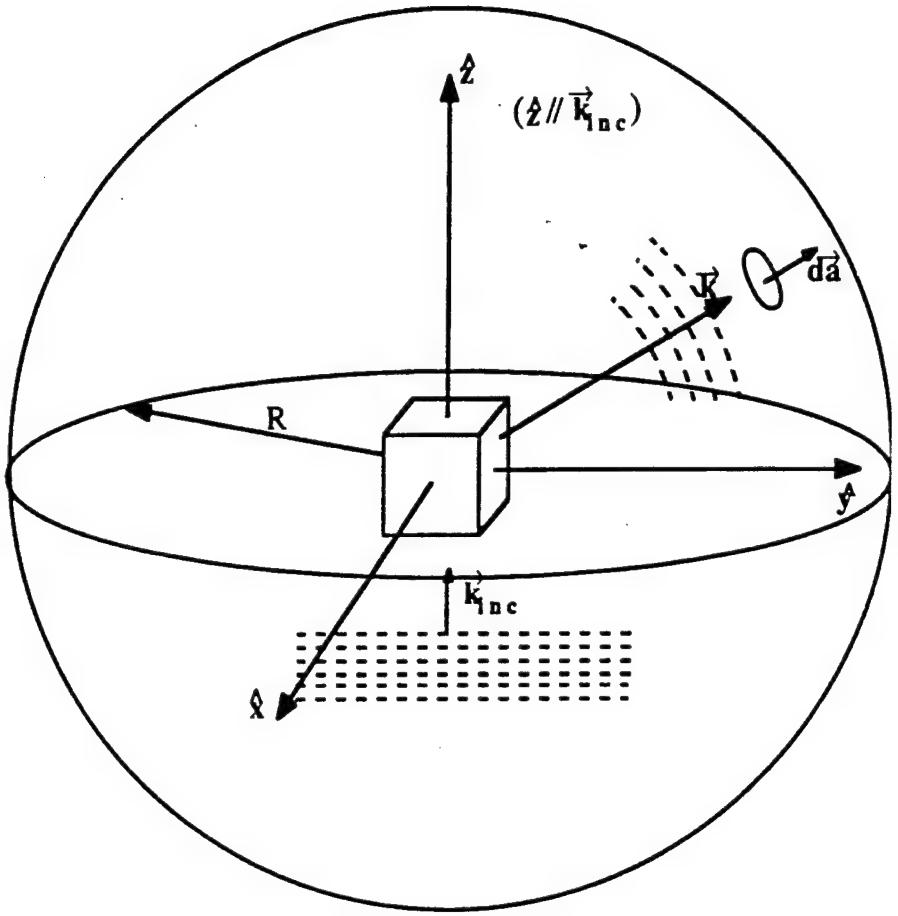

Figure 5. Scattering Geometry

We show a plane wave incident on the system from the bottom, with wave vector parallel to the z axis, and an outgoing wave incident on a small area element on a spherical shell, a distance R from the center of the scattering volume. The required factors are:

Area element: $d\vec{a} = R^2 \, d\Omega \, \hat{r}$,

Power: $P = \vec{S} \cdot \hat{k} \; R^2 \, d\Omega \quad \ldots (\hat{k} // \hat{r})$

field gradient: $\vec{\nabla}\pi = \vec{\nabla} \; \frac{e^{ikr}}{r} \; F(\vec{k}, \vec{k}_{inc})$

$$= F(\vec{k}, \vec{k}_{inc}) \left(ik \, \frac{e^{ikr}}{r} - \frac{e^{ikr}}{r^2} \right) \hat{r}$$

$$\cong ik \, F(\vec{k}, \vec{k}_{inc}) \, \frac{e^{ikr}}{r} \quad \ldots \text{(at large r , such as r} \sim \text{R,)}$$

On substituting these values into III.8.12, we obtain the time averaged power per unit area scattered in direction $\vec{k}$

$$\overline{\langle \vec{S} \rangle} = \frac{\omega\rho}{2} \operatorname{Im}\left(F^* \frac{e^{-ikR}}{R} i\vec{k} F \frac{e^{ikR}}{R} \right)$$

$$= \frac{\omega\rho}{2R^2} \vec{k} \overline{F^*F} \qquad \text{III.8.13}$$

Similarly we have for the incident flux

$$\overline{\langle \vec{S_{inc}} \rangle} = \frac{\omega\rho}{2} \operatorname{Im}\left(e^{-ik_{inc}z} i\vec{k}_{inc} e^{ik_{inc}z} \right)$$

$$= \frac{\omega\rho}{2} \vec{k}_{inc} \qquad \text{III.8.14}$$

On substituting these values into III.8.10, we obtain

$$\frac{d\sigma(\vec{k}, \vec{k}_{inc})}{d\Omega} = \overline{F^*F}, \qquad \text{III.8.15}$$

where $F = F(\vec{k}, \vec{k}_{inc})$ is the scattering amplitude.

Summary of Results for Scattering Cross Section

The final result of the scattering cross section calculation is the relation III.8.15

$$\frac{d\sigma(\vec{k}, \vec{k}_{inc})}{d\Omega} = \overline{F^*F},$$

where $F = F(\vec{k}, \vec{k}_{inc})$ is the scattering amplitude. The remaining task is to substitute the prior results derived from the self consistent field approximation, and evaluate these results. In particular, we have for the scattering amplitude, the expression III.8.9

$$F(\vec{k},\vec{k}_{inc}) = \frac{1}{4\pi} \oint_{S_1} e^{-i\vec{k}\cdot\vec{r}'} \hat{n}'\cdot [-i\vec{k}\,\pi_s(\vec{r}') - \nabla'\,\pi_s(\vec{r}')]da',$$

from which we can deduce that

$$\overline{F^*F} = \frac{1}{(4\pi)^2}\int_{S_1}\int_{S_1} e^{-i\vec{k}\cdot\vec{r}'} e^{i\vec{k}\cdot\vec{r}''} \times \overline{\hat{n}'\cdot[-i\vec{k} - \nabla']\pi_s(\vec{r}')\,\hat{n}''\cdot[i\vec{k} - \nabla'']\pi_s^*(\vec{r}'')}\,da'\,da''$$

$$= \frac{1}{(4\pi)^2}\int_{S_1}\int_{S_1} e^{-i\vec{k}\cdot(\vec{r}'-\vec{r}'')}\,\hat{n}'\cdot[-i\vec{k} - \nabla']\,\hat{n}''\cdot[i\vec{k} - \nabla'']\,\overline{\pi_s(\vec{r}')\pi_s^*(\vec{r}'')}\,da'\,da''$$

$$= \frac{1}{(4\pi)^2}\int_{S_1}\int_{S_1} e^{i\vec{k}\cdot(\vec{r}''-\vec{r}')} \times [\,\hat{n}'\cdot\vec{k}\;\hat{n}''\cdot\vec{k} + i\,\hat{n}'\cdot\vec{k}\;\hat{n}''\cdot\nabla'' - i\,\hat{n}'\cdot\nabla'\;\hat{n}''\cdot\vec{k} + \hat{n}'\cdot\nabla'\;\hat{n}''\cdot\nabla'']$$

$$\times\ \overline{\pi_s(\vec{r}')\pi_s^*(\vec{r}'')}\,da'\,da'', \qquad \text{III.8.16}$$

where we have interchanged the order of differentiation and averaging, since these refer to different variables.

The averaged product of the scattered waves can be written as

$$\overline{\pi_s(\vec{r}')\pi_s^*(\vec{r}'')} = \langle \vec{r}' | \overline{(|\pi_T\rangle - |p_i\rangle)(\langle \pi_T| - \langle p_i|)} | \vec{r}'' \rangle$$

$$= \langle \vec{r}' \overline{|\pi_T\rangle\langle \pi_T|} \vec{r}'' \rangle$$

$$- \langle \vec{r}' \overline{|\pi_T\rangle} \langle p_i| \vec{r}'' \rangle$$

$$- \langle \vec{r}' |p_i\rangle \overline{\langle \pi_T|} \vec{r}'' \rangle$$

$$+ \langle \vec{r}' |p_i\rangle\langle p_i| \vec{r}'' \rangle \qquad \text{III.8.17}$$

From III.7.11, we have to first order

$$\langle \vec{r}' | \overline{|\pi_T\rangle\langle \pi_T|} | \vec{r}'' \rangle = \langle \vec{r}' | (1 + L) \overline{|\pi_T\rangle} \overline{\langle \pi_T|} \vec{r}'' \rangle$$

$$= \langle \vec{r}' \overline{|\pi_T\rangle} \overline{\langle \pi_T|} \vec{r}'' \rangle + \langle \vec{r}' | L \overline{|\pi_T\rangle} \overline{\langle \pi_T|} \vec{r}'' \rangle$$

From III.7.9, we have

$$\langle \vec{r}' \overline{|\pi_T\rangle} = \sum_\sigma \frac{1}{1 - NT_\sigma} \langle \sigma | p_i\rangle \langle \vec{r}' | \sigma \rangle$$

and similarly,

$$\overline{\langle \pi_T|} \vec{r}'' \rangle = \left(\sum_\sigma \frac{1}{1 - NT_\sigma} \langle \sigma | p_i\rangle \langle \vec{r}'' | \sigma \rangle \right)^*.$$

From III.7.12, we have

$$\langle \vec{r}' | L \overline{|\pi_T\rangle} \overline{\langle \pi_T|} \vec{r}'' \rangle$$

$$= N \sum_{\sigma\sigma'\sigma''} \frac{T_{\sigma\sigma''} T^*_{\sigma\,\sigma''+\sigma'-\sigma}}{1 - NT_\sigma NT^*_{\sigma'}} \frac{\langle \sigma'' | p_i\rangle}{1 - NT_{\sigma''}} \frac{\langle \sigma''+\sigma'-\sigma | p_i\rangle}{1 - NT^*_{\sigma''+\sigma'-\sigma}} \langle \vec{r}' | \sigma \rangle \langle \sigma' | \vec{r}'' \rangle$$

The quantity $\langle \vec{r}' | \sigma \rangle$ is given by

$$\langle \vec{r}' | \sigma \rangle = \frac{1}{\sqrt{V}} e^{i\vec{k}_\sigma \cdot \vec{r}'},$$

and similarly,

$$\langle \sigma' | \vec{r}'' \rangle = \frac{1}{\sqrt{V}} e^{-i\vec{k}_{\sigma'} \cdot \vec{r}''}.$$

The quantity $\langle \sigma | p_i \rangle$ is given as

$$\langle \sigma | p_i \rangle = \int_V d^3r \ \langle \vec{k}_\sigma | \vec{r} \rangle \langle \vec{r} | \vec{k}_{inc} \rangle$$

$$= \frac{1}{V} \int_V d^3r \ e^{-i\vec{k}_\sigma \cdot \vec{r}} \ e^{i\vec{k}_{inc} \cdot \vec{r}}$$

$$= \frac{1}{V} \int_V d^3r \ e^{-i(\vec{k}_\sigma - \vec{k}_{inc}) \cdot \vec{r}}$$

An issue of importance is whether the incoming wave can be treated as a normal mode of the system. This will depend somewhat on the particular situation. If the wavelength of the incident wave is small compared to the total size of the turbulence distribution, then the density of modes in the vicinity of the incident wavevector will be high, and assumption that he incoming wave is a normal mode should give reasonable results. This condition is likely to be satisfied for the application of interest, for which case we may use the relationship

$$= \delta(\vec{k}_\sigma - \vec{k}_{inc}),$$

which significantly simplifies the analysis.

The matrix elements T_σ and $T_{\sigma\sigma'}$ from III.7.16 and III.7.19, are given as

$$T_\sigma = \frac{1}{\sqrt{V}} \sum_{n=0}^{\infty} c_{n_o} \int d^3r \ e^{-i k_\sigma r \cos\theta} \ h_n^{(1)}(k_\sigma r) \ P_n(\cos\theta),$$

and

$$T_{\sigma\sigma'} = \frac{1}{\sqrt{V}} \sum_{n=0}^{\infty} c_{n_o} \int d^3r \ e^{-i k_\sigma r \cos\gamma} \ h_n^{(1)}(k_\sigma r) \ P_n(\cos\theta).$$

Section III.9

Relationship to Tatarski's Approach

In this section we show how Tatarski's result for the average value of the flux density vector of the scattered energy (averaged over one period of oscillation) follows from our more general formulation by keeping only lowest order terms in the scattering potential. As in the first part of section III, we neglect the motion of the medium, for simplicity. Tatarski's equations (5.5), (5.8), (5.10), and (5.13) are then

(5.5)
$$(\nabla^2 + k^2)\,\pi = k^2 \frac{t'}{t}\,\pi \equiv v'\,\pi\,.$$

A series solution, $\pi = \pi_0 + \pi_1 + \ldots$, is then sought where

(5.8)
$$\pi_0 = A_0\, e^{i\,\vec{k}\cdot\vec{r}}$$

(5.10)
$$\pi_1 = -\frac{1}{4\pi}\,\frac{e^{ikr}}{r}\int_V 2\,k^2\,\frac{t'}{t}\,A_0\, e^{ik\hat{n}\cdot\vec{r}}\, e^{-ik\hat{m}\cdot\vec{r}}\, d^3r \qquad \text{(far field)}$$

The average scattered energy flux is given by

(5.13)
$$\vec{S} = \frac{\omega\rho}{2}\,\mathrm{Im}\,(\pi_1^{*}\nabla\pi_1)$$

$$= \hat{m}\,\frac{\rho c k^5 A_0^2}{8\pi^2 r^2}\int_V\int_V \overline{\frac{t'(\vec{r}_1)}{2t}\,\frac{t'(\vec{r}_2)}{2t}}\; e^{ik(\hat{n}-\hat{m})\cdot(\vec{r}_1-\vec{r}_2)}\, d^3r_1\, d^3r_2.$$

In our more abstract formalism, these same equations are written as

(5.5)
$$\hat{g}_0^{-1}\,|\pi\rangle = \hat{\Diamond}'\,|\pi\rangle \qquad \text{III.9.1}$$

(5.8)
$$|\pi_0\rangle\;; \qquad \text{with } \langle\,\vec{r}\,|\pi_0\rangle = A_0\, e^{i\,\vec{k}\cdot\vec{r}}$$

(5.10)
$$|\pi\rangle^{(1)} = \hat{g}_0\,\hat{\Diamond}'\,|\pi_0\rangle \qquad \text{(far field)} \qquad \text{III.9.2}$$

(5.13)
$$\vec{S} = \hat{g}_0\,\overline{\hat{\Diamond}'\,|\pi_0\rangle\langle\pi_0|\hat{\Diamond}'^{\dagger}}\,\hat{g}_0^{\dagger} \qquad \text{III.9.3}$$

As is evident from equation (5.10), or equivalently relation III.9.2, the solution for π is found to first order in the perturbing potential v', and the flux vector $\vec{S}$ is therefore obtained in the Born approximation. We will show that the relation III.9.3 for the average scattered energy flux follows from the results of our choice of multiple scattering formalism in the limit of neglecting the coherent scattering entirely and keeping the incoherent results only to lowest order. In particular, our results from section III.6 are

$$\overline{|\pi_T\rangle\langle\pi_T|} = (1 + L + L^2 + L^3 + \ldots)\ \overline{|\pi_T\rangle}\ \overline{\langle\pi_T|}$$

$$= (1 - L)^{-1}\ \overline{|\pi_T\rangle}\ \overline{\langle\pi_T|}, \qquad \text{III.9.4}$$

and

$$\overline{|\pi_T\rangle} = (1 + \overline{\hat{T}} + (\overline{\hat{T}})^2 + \ldots)\,|p_i\rangle$$

$$= (1 - \overline{\hat{T}})^{-1}\,|p_i\rangle, \qquad \text{III.9.5}$$

where

$$L = (1 - R)^{-1}\,(M) \text{ (for uncorrelated scaterers)}$$

$$= (1 + R + R^2 + R^3 + \ldots)\,M \qquad \text{III.9.6}$$

and

$$R\,X \equiv \overline{\hat{T}}\ X\ \overline{\hat{T}}^{\dagger} \qquad \text{III.9.7}$$

$$M\,X \equiv N \int d^3b_j\ \ p_1(\vec{b}_j)\ \ \hat{T}(\vec{b}_j)\ X\ \overline{\hat{T}}^{\dagger}(\vec{b}_j) = N\,\overline{\hat{T}X\hat{T}}, \qquad \text{III.9.8}$$

and

$$\hat{T} \equiv \hat{g}_o\,\hat{S} = \hat{g}_o\,\hat{v}'\,(1 - \hat{g}_o\,\hat{v}')^{-1}$$
$$= \hat{g}_o\,\hat{v}' + \hat{g}_o\,\hat{v}'\,\hat{g}_o\,\hat{v}' + \hat{g}_o\,\hat{v}'\,\hat{g}_o\,\hat{v}'\,\hat{g}_o\,\hat{v}' + \ldots \qquad \text{III.9.9}$$

To get to Tatarski's first Born approximation result, we take the limit of $\hat{V}'$ going to zero, keeping only the lowest power non vanishing terms in each expansion. In particular, this means that we take $\hat{T}$ to be

$$\hat{T} \equiv \hat{g}_o \hat{V}' . \qquad \text{III.9.10}$$

In addition, we find III.9.5 is replaced by

$$\overline{|\pi_T\rangle} \equiv |p_i\rangle ,$$

so that

$$\overline{|\pi_T\rangle}\ \overline{\langle \pi_T|} \equiv |p_i\rangle\langle p_i| . \qquad \text{III.9.11}$$

We may now evaluate

$$\overline{|\pi_T\rangle\langle\pi_T|} = (1-L)^{-1}\,\overline{|\pi_T\rangle}\ \overline{\langle\pi_T|} \equiv (1+L)\,\overline{|\pi_T\rangle}\ \overline{\langle\pi_T|},$$

where we must now consider L to be given by $L \equiv M$, where

$$MX \equiv \overline{\hat{g}_o \hat{V}'\, X\, \hat{V}'^{\dagger} \hat{g}_o^{\dagger}} .$$

If we use the result III.9.11, we find

$$\overline{|\pi_T\rangle\langle\pi_T|} \equiv |p_i\rangle\langle p_i| + \overline{\hat{g}_o \hat{V}'\ |p_i\rangle\langle p_i|\ \hat{V}'^{\dagger}\hat{g}_o^{\dagger}} . \qquad \text{III.9.12}$$

If we now consider the scattered component only, we find to this order of approximation,

$$\overline{|\pi_s\rangle\langle\pi_s|} \equiv \overline{|\pi_T\rangle\langle\pi_T|} - \overline{|\pi_T\rangle}\ \langle p_i| - |p_i\rangle\,\overline{\langle\pi_T|} + |p_i\rangle\langle p_i|$$

$$\equiv \overline{\hat{g}_o \hat{V}'\ |p_i\rangle\langle p_i|\ \hat{V}'^{\dagger}\hat{g}_o^{\dagger}} , \qquad \text{III.9.13}$$

which is in agreement with Tatarski's result.

Sction III.10

Numerical Algorithm

The numerical algorithm which is presented in this section is a fully operational code for obtaining the differential scattering cross section in the mean self consistent field approximation for a collection of randomly positioned eddies confined to a cubical turbulence volume. The code follows the theoretical treatment developed in sections III.1.3 through III.1.8 of this document. Both the computation of the coherent field and the incoherent field have been coded. The incoherent field portion would lend itself quite well to operation as a vectorized code. It does, however, involve an extra three dimensional momentum space integration, and operates very slowly on the pc computer which has been used for code development . For this reason, the incoherent field portion of the algorithm has been commented out in the phase I algorithm. This is clearly marked by the use of a "c*" notation in the first column of the listing. It could easily be reinstated for evaluation on a vector processor, if desired.

The treatment in the numerical algorithm makes use of a code segment obtained from a text by Barber and Hill, for computation of the Bessel function expansions identified in the T-matrix plane wave representation portion of part III of this report. It does, however, diverge from the coordinate system choice identified in section III.7 of the theory. This is because an election was made to treat a definite cubical turbulence volume, for which case there is a preferred orientation of the spacial coordinates along the axes of the cube, rather than choosing the z axis aligned to the k-space wave vector. The angle between the space vector and the wave vector is still computed, and the result is used in the spherical expansion of the single particle solution. The only signigicant consequence of the coordinate system orientation choice is that the volume integrals are carried out in cartesian, rather than polar, coordinates.

The selection of a cube for the shape of the turbulence volume is arbitrary, but does lend itself to computational simplicity. There is, at this stage of code development, no specified a priori shape to associate with the physical turbulence volume. Shapes of a rectangular prism character, but with non cubical aspect ratio could be computed with negligible perturbation to the algorithm as it stands; many of the required code lines for this modification have already been inserted, but variables in these lines have been set to default cubical values.

The treatment of an arbitrarily shaped turbulence volume lies outside the scope of phase I development, but should be relatively straightforward to accomplish under a phase II. In particular, a natural way to approach the task would be to approximate the volume with a collection of rectangular prisms, joined at the boundaries by transport boundary conditions. This would allow preservation of the main features of the phase I code. The

process of joining adjacent prisms would be very similar in character to the processes used in finite element computational fluid dynamics algorithms.

Some compromises have been made in the development of the phase I algorithm, in the interest of both schedule and run time. In particular, the code as presented treats only a single turbule size. The treatment of a collection of sizes is a straight forward modification in the T-matrix calculkations, in which the averaging of the collection of waves to obtain a mean wave would yield not a simple multiplication factor of N, but rather a sum over sizes of the product of N for each size with the quantity being averaged. The resulting T matrices would have additional indices for the turbule size. An algorithm for the computation of the number densities for each size, within the confines of a particular turbulence model, has been presented in Part II of this report, but has not yet been integrated into the scattering algorithm. We have also developed a code module for inverting the matrix which results from the part II algorithm when a large number of size bins have been selected. This code is also operational, but has not yet been fully tested, and is not presented in this phase I report. It is suggested that the phase II code development should include as a first step the integration of these two codes with the body of the algorithm, and the development of the T-matrices when there is more than one size.

Use of Algorithm for Shadow Zone Analysis

Several options present themselves for the intended use of this code in shadow zone analysis. One of these options is to use the algorithm as is, but embed it in a larger code which has been designed to simulate the shadow zone environment. Some characteristics of the code should be identified in the assessment of the viability of this option. First, the election to use a cubical turbulence volume will lead to volume boundary artifacts when the scatterer density is high, because the planar surfaces will behave as mirrors. This is probably not a significant issue at low eddy density, because in that case, the acoustic wave would penetrate well into the volume, and the scattered wave would be principally influenced by volumetric, rather than surface effects. Another consideration is that a far field approximation has been used. This will be highly appropriate when the turbulence volume is small and reasonably distant. If the smallness condition is not satisfied, but the turbulence volume is still reasoonably distant, it may be approprate to subdivide the volume into smaller regions, and compute the contribution from each within the far field approximation. This procedure will be invalid for wavelengths which are comparable to the size of the total turbulence volume, but such wavelengths are not likely to characterize the acoustic spectrum of interest , since this lies in the midrange audio spectrum.

A second option in the use of the code in shadow zone analysis would be to conduct the additional development required to handle the near field. This would allow characterization of scatter from turbule distributions with a large angular subtense.

Additional theoretical development would be required before a code module of this character could be written.

Some consideration should be given as to how to interface the current module with a system level environmental description model. One method for treating the system level model would be essentially a finite element fluid model with acoustic transport. If this style of code is envisioned, the use of the current module would be straightforward in that one or more of the finite elements would simply be replaced by the current algorithm in so far as that elements acoustic properties are concerned. If the system level algorithm is to be a more simplified description, such a s a linear circuit model, some thought would need to be applied as to how to represent this current algorithm as an equivalent circuit element.

Finally, if the goal is to obtain as literal as possible an environmental description, it would be desirable to extend the current algorithm to arbitrary shapes of the total turbulence volume. This would involve analysis of interface acoustic transport equations, and the encoding of these equations between rectangular elements. A Finite Volume Elements approach would lend itself well to this task. As an alternative, it might be desired to approximate the actual physical turbulence volume by some relatively simple three dimensional shape of high symmetry, such as a truncated cylinder, an oblate spheroid, etc. In this latter case, some additional development work would be required to recast the integrals into the selected geometry.

Phase I algorithm

The text of the phase I algorithm is presented below.

```
      program acstc
      complex hankel(200),hprime(200),xi,
     +        b(200),c(200),e(200),f(200),cn0(200),dum
      common /avkncom/ N,Tix,Tiy,den
      common /tcom/ el,nmax,pi,jmax,cn0
      dimension xk(3),xki(3),a(200),d(200)
c*      dimension xksg(3),isg(3)
c  N is the number of eddies in the volume
c  Tix,Tiy are the real and imaginary parts of the T matrix
c  den is an intermediate expression used in evaluating T
c  el is the side length of the (assumed cubical) turbulence volume
c  nmax is the highest order to use in spherical expansions
c  jmax is the number of cells per side, when dividing up the
c    volume for doing volume integrals.
      pi = 3.141592653
c  hankel(n) is a complex array for storing hankel functions
c
```

```
c  open input data file
      open(1,file='acstc.inp')
      read(1,*) maxsg,el,(xk(i),i=1,3),(xki(j),j=1,3),N,jmax,nmax
c
c  open output file
      open(7,file='acstc.out')
c
c  specify the range of allowed values of sigma to compute
      maxsg1 = maxsg
      maxsg2 = maxsg
      maxsg3 = maxsg
c
c  evaluate the coefficients to be used in spherical expansions
c
c ...first some assumptions, to use as a definite example...
c
      xa = 10.
c    xa is the radius of turbule (cm)
c
      xkimag=sqrt(xki(1)*xki(1)+xki(2)*xki(2)+xki(3)*xki(3))
c    xkimag is the scalar magnitude of the incident wave vector
c
      rho = .0018
c    rho is the density of air (g/cm^3) outside eddy
c
      rhoprm = rho*1.0001
c    rhoprm is the density inside the eddy
c    (assume a small (0.01 percent) density fluctuation)
c
      xkprm = 1.0002*xkimag
c    xkprm is the magnitude of the wave vector inside the turbule
c    (assume a small (0.02 percent) refractive index fluctuation)
c
c    note: variations in rho and n may or may not be related,
c      depending on temperature, speed, moisture, etc considerations
c
c  The coefficients cn0 can be found by solving the relations
c            a*an0 - b*cn0 = c
c  and         d*an0 - e*cn0 = f
c  with the coefficients a through f given below
c
      xi = (0.,1.)
c    xi is the imaginary number square root of -1.
c
c  evaluate the required (exterior) bessel functions at the boundary
      write(6,*) ' start 1st besh call'
      write(7,*) ' start 1st besh call'
      call besh(xkimag*xa,hankel,nmax+1)
c
c  evaluate the required derivatives of bessel functions at boundary
      write (6,*) ' return from besh... start derivatives'
      write (7,*) ' return from besh... start derivatives'
      hprime(1) = - hankel(2)
      do 40 i=2,nmax+1
```

```
40     hprime(i) = hankel(i-1) - (i+1)*hankel(i)/xkimag/xa
c
c  write the exterior coefficients in terms of bessels and derivatives
       write(6,*) ' done with derivatives, start exterior coefficients'
       write(7,*) ' done with derivatives, start exterior coefficients'
       do 50 i=1,nmax
       ip = i+1
       b(i) = rho*hankel(ip)
       c(i) = rho*(1.+2.*i)*xi**i*real(hankel(ip))
       e(i) = xkimag*hprime(ip)
50     f(i) = xkimag*(1.+2.*i)*xi**i*real(hprime(ip))
c
c  evaluate the required (interior) bessel functions at boundary
       write(6,*) ' done with exterior...do interior besh call'
       write(7,*) ' done with exterior...do interior besh call'
       call besh(xkprm*xa,hankel,nmax+1)
c
c  evaluate the required derivatives of bessel functions at boundary
       write(6,*) ' done with besh, interior start derivatives'
       write(7,*) ' done with besh, interior start derivatives'
       hprime(1) = - hankel(2)
       do 60 i=2,nmax+1
60     hprime(i) = hankel(i-1) - (i+1)*hankel(i)/xkprm/xa
c
c  write the interior coefficients in terms of bessels and derivatives
       write(6,*) ' evaluate interior coefficients and solve for cn0'
       write(7,*) ' evaluate interior coefficients and solve for cn0'
       do 70 i=1,nmax
       ip = i+1
       a(i) = rhoprm*real(hankel(ip))
       d(i) = xkprm*real(hprime(ip))
c
c  solve for the cn0
       dum = b(i)*d(i)-e(i)*a(i)
       if(dum.eq.(0.,0.))go to 99
70     cn0(i)=(f(i)*a(i)-c(i)*d(i))/dum
c
c  compute first term of dsigma/domega
       write(6,*) ' done with cn0... start dsigma/domega #1'
       write(6,*) 'cn0 = ',(cn0(i),i=1,nmax)
       write(7,*) 'cn0 = ',(cn0(i),i=1,nmax)
       write(7,*) ' done with cn0... start dsigma/domega #1'
       fi1 = el*el*el*avkna(xki)*fi(xk,xki)/16./pi/pi
       write(6,*) 'fi1=',fi1
       write(7,*) 'fi1=',fi1
c
cc*  compute second term of dsigma/domega
cc*   sum fi2 over ksigma for allowed values of sigma
c*       write(6,*) ' now do dsigma/domega #2'
c*       write(7,*) ' now do dsigma/domega #2'
       fi2 = 0.
c*       do 300 i1 = -maxsg1, maxsg1
c*       isg(1) = i1
c*       do 200 i2 = -maxsg2, maxsg2
```

```
c*       isg(2) = i2
c*       do 100 i3 = -maxsg3, maxsg3
c*       isg(3) = i3
cc*   compute ksigma
c*       do 400 i=1,3
c*400     xksg(i) = 2*pi*isg(i)/el
c*       fi2 = fi2+el*el*el*avkns(xksg,xki)*fi(xk,xksg)/16/pi/pi
c*       write(6,*) ' done with avkns for ksigma =',xksg
c*       write(7,*) ' done with avkns for ksigma =',xksg
c*100    continue
c*200    continue
c*300    continue
c
c  combine first and second term of dsigma/domega
c*       write(6,*) ' add fi1 and fi2 to get total dsigma/domega'
c*       write(7,*) ' add fi1 and fi2 to get total dsigma/domega'
      fstrf = fi1 + fi2
      write(6,*) 'fstrf = ',fstrf
      write(7,*) 'fstrf = ',fstrf
c
c  all done,
      stop
99     write(6,*) ' boundary matrix is singular'
      stop
      end
c-----------------------------------------------------------------------
      function fi(xk1,xk2)
      dimension xk1(3),xk2(3),smallq(3),bigQ(3)
      write(6,*) ' enter function fi'
      xf = 1.
      qdotQ = 0.
      do 100 i=1,3
      smallq(i) = xk1(i) - xk2(i)
      bigQ = xk1(i) + xk2(i)
      qdotQ = qdotQ + smallq(i) * bigQ(i)
      xf = xf*xg(smallq(i))
100    continue
      fi = qdotQ*qdotQ*xf
      write(6,*) ' done with function fi'
      return
      end
c-----------------------------------------------------------------------
      function xg(x)
      write(6,*) ' start function xg'
      if(x.le.0.001)go to 10
      xg = sin(x)/x*sin(x)/x
      write(6,*) ' done with xg'
      return
10     xg = (1.-x*x/6.)*(1.-x*x/6.)
      write(6,*) ' done with xg'
      return
      end
c-----------------------------------------------------------------------
      function avkna(xki)
```

```
      common /avkncom/ N,Tix,Tiy,den
      dimension xki(3)
      write(6,*) ' start function avkna',' xki=',xki
      call T(xki,Tix,Tiy)
      den = (1.- N*Tix)*(1.- N*Tix) + N*N*Tiy*Tiy
      avkna = N*N*(Tix*Tix + Tiy*Tiy)/den
      write(6,*) ' done with avkna'
      return
      end
c-----------------------------------------------------------------------
      function avkns(xksg,xki)
      common /avkncom/ N,Tix,Tiy,den
      dimension xksg(3),xki(3)
      write(6,*) ' start function avkns'
      write(6,*) ' xki = ',xki
      write(7,*) ' xki = ',xki
      call T(xksg,Tsgx,Tsgy)
      write(7,*) ' Tsgx = ',Tsgx,'Tsgy = ',Tsgy
      den2 = 1.- N*N*(Tsgx*Tsgx + Tsgy*Tsgy)
      call T2(xksg,xki,T2sgx,T2sgy)
      avkns = N*(T2sgx*T2sgx + T2sgy*T2sgy)/den2/den
      write(6,*) ' done with avkns'
      return
      end
c-----------------------------------------------------------------------
      subroutine T(xk,Tx,Ty)
      common /Tcom/ el,nmax,pi,jmax,cn0
      complex sphsum,hn(200),Texp,TT,cn0(200)
      dimension xk(3),pn(200)
c  xk is the wavevector
c  Tx is the real part of TT
c  Ty is the imaginary part of TT
c  el is the side length of the cubical turbulence volume
c  nmax is the cut off for the summation of spherical functions
c  jmax is the # of cells into which each side of the cube is divided
      write(6,*) ' enter subroutine T',' xk=',xk
      write(6,*) ' establish limits of r integration'
      nxmax = jmax
      nymax = jmax
      nzmax = jmax
      write(6,*) ' set d-cubed r'
      deltax = el/nxmax
      deltay = el/nymax
      deltaz = el/nzmax
      write(6,*) ' set starting r'
      x0 = (nxmax + 1)*deltax/2.
      y0 = (nymax + 1)*deltay/2.
      z0 = (nzmax + 1)*deltaz/2.
      write(6,*) ' find magnitude of k vector'
      xkmag = sqrt(xk(1)*xk(1) + xk(2)*xk(2) + xk(3)*xk(3))
      TT = (0.,0.)
      write(6,*) 'start r integration'
      do 400 iz=1,nzmax
      z = iz*deltaz - z0
```

```
      z2 = z*z
      write(6,*) ' z=',z,' z2=',z2
      do 300 iy=1,nymax
      y = iy*deltay - y0
      y2 = y*y
      write(6,*) ' y=',y,' y2=',y2
      do 200 ix=1,nxmax
      x = ix*deltax - x0
      x2 = x*x
      write(6,*) ' x=',x,' x2=',x2
      r = sqrt(x2 + y2 + z2)
      write(6,*) ' r=',r
      rdotk = (x*xk(1) + y*xk(2) + z*xk(3))
      write(6,*) ' rdotk=',rdotk
      rk = r*xkmag
      write(6,*) ' rk=',rk
      cosang = rdotk/rk
      write(6,*) ' cosang=',cosang
c  cosang is the cosine of the angle between r and k vectors
      sphsum = (0.,0.)
      write(6,*) ' doing r',ix,iy,iz
      write(6,*) ' start subroutine p'
      call p(cosang,pn,nmax)
      write(6,*) ' done with subroutine p, enter subroutine besh'
      call besh(rk,hn,nmax+1)
      write(6,*) ' done with besh... start sphsum integral'
      do 100 n=1,nmax
100     sphsum = sphsum + cn0(n)*hn(n+1)*pn(n)
      Texp = cmplx(cos(rdotk),-sin(rdotk))
200     TT = TT + Texp*sphsum*deltax*deltay*deltaz
300     continue
400     continue
      TT = TT/el**1.5
      Tx = real(TT)
      Ty = aimag(TT)
      write(6,*) ' done with subroutine T'
      return
      end

      subroutine T2(xk1,xk2,Tx,Ty)
      common /Tcom/ el,nmax,pi,jmax,cn0
      dimension xk1(3),xk2(3),pn(200)
      complex sphsum,hn(200),Texp,T,cn0(200)
c  xk1 is the "outgoing" wavevector
c  xk2 is the "incoming" wavevector
c  Tx is the real part of T2
c  Ty is the imaginary part of T2
c  el is the side length of the cubical turbulence volume
c  nmax is the cut off for the summation of spherical functions
c  jmax is the # of cells into which each side of the cube is divided
      nxmax = jmax
      nymax = jmax
      nzmax = jmax
      deltax = el/nxmax
```

```
      deltay = el/nymax
      deltaz = el/nzmax
      x0 = (nxmax + 1)*deltax/2.
      y0 = (nymax + 1)*deltay/2.
      z0 = (nzmax + 1)*deltaz/2.
      xk1mag=sqrt(xk1(1)*xk1(1)+xk1(2)*xk1(2)+xk1(3)*xk1(3))
      xk2mag=sqrt(xk2(1)*xk2(1)+xk2(2)*xk2(2)+xk2(3)*xk2(3))
      T = (0.,0.)
      do 400 iz=1,nzmax
      z = iz*deltaz - z0
      z2 = z*z
      do 300 iy=1,nymax
      y = iy*deltay - y0
      y2 = y*y
      do 200 ix=1,nxmax
      x = ix*deltax - x0
      x2 = x*x
      r = sqrt(x2 + y2 + z2)
      rdotk1 = (x*xk1(1) + y*xk1(2) + z*xk1(3))
      rdotk2 = (x*xk2(1) + y*xk2(2) + z*xk2(3))
      rk1 = r*xk1mag
      rk2 = r*xk2mag
      cosgam = rdotk1/rk1
      costht = rdotk2/rk2
c costht is the cosine of the angle between r and k2 vectors
c cosgam is the cosine of the angle between r and k1 vectors
      sphsum = (0.,0.)
      call p(costht,pn,nmax)
      call besh(rk2,hn,nmax+1)
      do 100 n=1,nmax
100     sphsum = sphsum + cn0(n)*hn(n+1)*pn(n)
      Texp = cmplx(cos(rdotk1),-sin(rdotk1))
200     T =Texp*sphsum*deltax*deltay*deltaz
300     continue
400     continue
      T = T/el**1.5
      Tx = real(T)
      Ty = aimag(T)
      return
      end

      Subroutine besh(x,hankel,nc)
c   This subroutine taken from Barber and Hill, 'Light
c   Scattering by Particles: Computational Methods,'
c   World Scientific, vol2, 1990
c
c     calculates hankel function
c     bj = bessel function of first kind
c     by = bessel function of second kind
c     x  = real argument
c     nc = order of functions... starts at zero... ie:
c          hankel(1) = (j0,y0); hankel(2) = (j1,y1); etc.
c
      complex hankel(nc)
```

```
      dimension bj(201),by(201),t(3)
c
c  by(*) calculation... obtain 0th & 1st order functions
      a = sin(x)/x
      by(1) = -cos(x)/x
      by(2) = by(1)/x - a
c
c  obtain the higher order functions by upward recursion
      do n = 3,nc
        rn = real(n-2)
        by(n) = (2.*rn+1.)*by(n-1)/x-by(n-2)
      end do
c
c  bj(*) calculation... set the starting order for downward recursion
      nst = int(x+4.05*x**.3333+2.+(101.+x)**.5)
c  the t(*) array is used to recur down to the two highest order functions
c  that are needed.
c
c  set starting values for the two highest orders nst and nst-1
      t(3)=0.
      t(2)=1.e-35
c
c  recur downward to obtain orders nc-1 and nc-2
      do i = nst-1,nc-1,-1
        ri = real(i)
        t(1) = (2.*ri+1.)*t(2)/x-t(3)
        t(3) = t(2)
        t(2) = t(1)
      end do
c
c  continue downward recursion to order one
      bj(nc) = t(3)
      bj(nc-1) = t(2)
      do i = nc-2,1,-1
        ri = real(i)
        bj(i) = (2.*ri+1.)*bj(i+1)/x-bj(i+2)
      end do
c
c  calculate the scale factor and the functions
      alpha = a/bj(1)
      do k = 1,nc
        hankel(k) = cmplx(bj(k)*alpha,by(k))
      end do
      return
      end

      subroutine p(x,pn,nmax)
      dimension pn(200)
      pn(1) = x
      pn(2) = 0.5*(3.*x*x-1.)
      do 10 i=3,nmax
10      pn(i) = ((2.*i-1.)*x*pn(i-1) - (i-1)*pn(i-2))/i
      return
      end
```

Appendix A

First Order Incoherent Effects

We have shown in section III.6 that the incoherent radiation component is given by

$$\overline{|\pi_T\rangle\langle\pi_T|} = (\mathbf{1} - L)^{-1}\ \overline{|\pi_T\rangle}\ \ \overline{\langle\pi_T|}\ ,$$

with

$$L = (\mathbf{1} - R)^{-1}(M + Q - R),$$

where M represents the contribution due to individual scatterers and $Q - R$ represents the contribution due to two particle correlation effects among pairs of scatterers. Within the context of our phase I assumptions, correlation effects have been deferred to the future, so that we have

$$L = (\mathbf{1} - R)^{-1} M\ . \qquad \text{A.1}$$

The superoperators R and M are defined for any configuration independent operator X as

$$R\ X \equiv \overline{\hat{T}}\ X\ \overline{\hat{T}}^{\dagger} \qquad \text{A.2}$$

and

$$M\ X \equiv \sum_j \int d^3r_j\, p(\vec{r_j})\, \hat{T}(\vec{r_j})\, X\, \hat{T}^{\dagger}(\vec{r_j})\,. \qquad \text{A.3}$$

The averaged operator $\overline{\hat{T}}$ is given in relation III.6.5 as

$$\overline{\hat{T}} = \sum_j \int d^3r_j\, p(\vec{r_j})\, \hat{T}(\vec{r_j}),$$

and has the plane wave expansion III.7.6

$$\overline{\hat{T}} = N \sum_{\sigma} T_{\sigma}\, |\sigma\rangle\langle\sigma|\,. \qquad \text{A.4}$$

The expression $(\mathbf{1} - R)^{-1} M$ can be expressed in the plane wave basis by expanding the operator in a series, and evaluating the separate terms of the series. The result is the

combined with the plane wave expansion of M. The first term in the series is X itself, which can be represented in the plane wave basis as

$$\mathbf{1}\,X\,\mathbf{1} = \sum_{\sigma}\sum_{\sigma'} |\sigma\rangle\langle\sigma|X|\sigma'\rangle\langle\sigma'| \equiv \sum_{\sigma}\sum_{\sigma'} X_{\sigma\sigma'}\,|\sigma\rangle\langle\sigma'|.$$

The second term in the series is $R\ X$, which can be expressed in the plane save basis as

$$R\ X \equiv \overline{\hat{T}}\ X\ \overline{\hat{T}}^{\dagger} = \sum_{\sigma}\sum_{\sigma'} (NT_{\sigma})\,|\sigma\rangle\langle\sigma|X|\sigma'\rangle\langle\sigma'|\,(NT_{\sigma'}{}^{*})$$

$$= \sum_{\sigma}\sum_{\sigma'} (NT_{\sigma})\,(NT_{\sigma'}{}^{*})\,X_{\sigma\sigma'}\,|\sigma\rangle\langle\sigma'|. \qquad \text{A.5}$$

The third term in the series is

$$R^{2}\,X = R\,(R\ X). \qquad \text{A.6}$$

$$= \sum_{\sigma}\sum_{\sigma'} (NT_{\sigma})\,|\sigma\rangle\langle\sigma|(R\ X)|\sigma'\rangle\langle\sigma'|(NT_{\sigma'}{}^{*}) \qquad (\text{...from A.5})$$

$$= \sum_{\sigma}\sum_{\sigma'} (NT_{\sigma})\,|\sigma\rangle\langle\sigma|\sum_{\sigma''}\sum_{\sigma'''} (NT_{\sigma''})\,(NT_{\sigma'''}{}^{*})\,X_{\sigma''\sigma'''}\,|\sigma''\rangle\langle\sigma'''|\,\sigma'\rangle\langle\sigma'|(NT_{\sigma'}{}^{*})$$

$$= \sum_{\sigma}\sum_{\sigma'}\sum_{\sigma''}\sum_{\sigma'''} (NT_{\sigma''})\,(NT_{\sigma'''}{}^{*})\,(NT_{\sigma})\ (NT_{\sigma'}{}^{*})\,X_{\sigma''\sigma'''}\,|\sigma\rangle\langle\sigma|\sigma''\rangle\langle\sigma'''|\sigma'\rangle\langle\sigma'| \qquad \text{A.7}$$

$$= \sum_{\sigma}\sum_{\sigma'}\sum_{\sigma''}\sum_{\sigma'''} (NT_{\sigma''})\,(NT_{\sigma'''}{}^{*})\,(NT_{\sigma})\ (NT_{\sigma'}{}^{*})\,X_{\sigma''\sigma'''}\,|\sigma\rangle\,\delta_{\sigma\sigma''}\,\delta_{\sigma'''\sigma'}\langle\sigma'|$$

$$= \sum_{\sigma}\sum_{\sigma'} (NT_{\sigma})^{2}\,(NT_{\sigma'}{}^{*})^{2}\,X_{\sigma\sigma'}\,|\sigma\rangle\langle\sigma'|. \qquad \text{A.8}$$

By induction, then,

$$R^{m}\,X = R\,(R^{m-1}X)$$

$$= \sum_{\sigma}\sum_{\sigma'} (NT_{\sigma})\,|\sigma\rangle\langle\sigma|(R^{m-1}X)|\sigma'\rangle\langle\sigma'|(NT_{\sigma'}{}^{*})$$

$$= \sum_{\sigma}\sum_{\sigma'} (NT_{\sigma})^{m}\,(NT_{\sigma'}{}^{*})^{m}\,X_{\sigma\sigma'}\,|\sigma\rangle\langle\sigma'|. \qquad \text{A.9}$$

On combining the terms in the sum we get

$$(\mathbf{1}-R)^{-1}X = \sum_{\sigma}\sum_{\sigma'}\left[1 + (NT_{\sigma})(NT_{\sigma'}^{*}) + (NT_{\sigma})^{2}(NT_{\sigma'}^{*})^{2} + \ldots + (NT_{\sigma})^{m}(NT_{\sigma'}^{*})^{m} + \ldots\right] X_{\sigma\sigma'}\,|\sigma\rangle\langle\sigma'|$$

$$= \sum_{\sigma}\sum_{\sigma'}\frac{1}{1-(NT_{\sigma})(NT_{\sigma'}^{*})}\,X_{\sigma\sigma'}\,|\sigma\rangle\langle\sigma'|. \qquad \text{A.10}$$

In order to get $L\,X = (\mathbf{1}-R)^{-1}M\,X$, we need next to compute $M\,X$ in the plane wave basis:

$$M\,X = \sum_{j}\int d^{3}r_{j}\,p(\vec{r}_{j})\,\hat{T}(\vec{r}_{j})\,X\,\hat{T}^{\dagger}(\vec{r}_{j})$$

$$= N\,\frac{1}{V}\int d^{3}r_{j}\;\mathbf{1}\,\hat{T}(\vec{r}_{j})\,\mathbf{1}\,X\,\mathbf{1}\,\hat{T}^{\dagger}(\vec{r}_{j})\,\mathbf{1}. \qquad \text{A.11}$$

The integrand of this expression is

$$\mathbf{1}\,\hat{T}(\vec{r}_{j})\,\mathbf{1}\,X\,\mathbf{1}\,\hat{T}^{\dagger}(\vec{r}_{j})\,\mathbf{1} =$$

$$= \sum_{\sigma_1}\sum_{\sigma_2}\sum_{\sigma_3}\sum_{\sigma_4}|\sigma_1\rangle\langle\sigma_1|\,\hat{T}(\vec{r}_{j})\,|\sigma_2\rangle\langle\sigma_2|X\,|\sigma_3\rangle\langle\sigma_3|\,\hat{T}^{\dagger}(\vec{r}_{j})\,|\sigma_4\rangle\langle\sigma_4|. \qquad \text{A.12}$$

$$= \sum_{\sigma_1}\sum_{\sigma_2}\sum_{\sigma_3}\sum_{\sigma_4}|\sigma_1\rangle e^{-i(\vec{k}_{\sigma_1}-\vec{k}_{\sigma_2})\cdot\vec{r}_{j}}\,T_{\sigma_1\sigma_2}\,X_{\sigma_2\sigma_3}\,T^{*}_{\sigma_4\sigma_3}\,e^{-i(\vec{k}_{\sigma_3}-\vec{k}_{\sigma_4})\cdot\vec{r}_{j}}\langle\sigma_4| \qquad \text{A.13}$$

$$= \sum_{\sigma_1}\sum_{\sigma_2}\sum_{\sigma_3}\sum_{\sigma_4}e^{-i(\vec{k}_{\sigma_1}-\vec{k}_{\sigma_2}+\vec{k}_{\sigma_3}-\vec{k}_{\sigma_4})\cdot\vec{r}_{j}}\,T_{\sigma_1\sigma_2}\,T^{*}_{\sigma_4\sigma_3}\,X_{\sigma_2\sigma_3}\,|\sigma_1\rangle\langle\sigma_4| \qquad \text{A.14}$$

so that the integral becomes

$$M\,X = \sum_{j}\int d^{3}r_{j}\,p(\vec{r}_{j})\,\hat{T}(\vec{r}_{j})\,X\,\hat{T}^{\dagger}(\vec{r}_{j}) =$$

$$\frac{N}{V}\int d^{3}r_{j}\sum_{\sigma_1}\sum_{\sigma_2}\sum_{\sigma_3}\sum_{\sigma_4}e^{-i(\vec{k}_{\sigma_1}-\vec{k}_{\sigma_2}+\vec{k}_{\sigma_3}-\vec{k}_{\sigma_4})\cdot\vec{r}_{j}}\,T_{\sigma_1\sigma_2}\,T^{*}_{\sigma_4\sigma_3}\,X_{\sigma_2\sigma_3}\,|\sigma_1\rangle\langle\sigma_4|$$

$$= N\sum_{\sigma_1}\sum_{\sigma_2}\sum_{\sigma_3}\sum_{\sigma_4}\delta_{\sigma_1+\sigma_3,\sigma_2+\sigma_4}\,T_{\sigma_1\sigma_2}\,T^{*}_{\sigma_4\sigma_3}\,X_{\sigma_2\sigma_3}\,|\sigma_1\rangle\langle\sigma_4| \qquad \text{A.15}$$

$$= N\sum_{\sigma_1}\sum_{\sigma_2}\sum_{\sigma_4}T_{\sigma_1\sigma_2}\,T^{*}_{\sigma_4(\sigma_2+\sigma_4-\sigma_1)}\,X_{\sigma_2(\sigma_2+\sigma_4-\sigma_1)}\,|\sigma_1\rangle\langle\sigma_4| \qquad \text{A.16}$$

To continue the derivation of $L\ X = (1 - R)^{-1} M\ X$, we substitute $X \rightarrow M\ X$ in relation A.10. To accomplish this, we need to evaluate the matrix elements of $M\ X$ which we derive from relation A.16:

$$\langle \sigma | M\ X | \sigma' \rangle = N \sum_{\sigma_1} \sum_{\sigma_2} \sum_{\sigma_4} T_{\sigma_1\sigma_2} T^*_{\sigma_4(\sigma_2+\sigma_4-\sigma_1)} X_{\sigma_2(\sigma_2+\sigma_4-\sigma_1)} \langle \sigma | \sigma_1 \rangle \langle \sigma_4 | \sigma' \rangle$$

$$= N \sum_{\sigma_2} T_{\sigma\sigma_2} T^*_{\sigma'(\sigma_2+\sigma'-\sigma)} X_{\sigma_2(\sigma_2+\sigma'-\sigma)} \qquad \text{A.17}$$

We next substitute this value into A.10, obtaining

$$L\ X = (1 - R)^{-1} M\ X = \sum_{\sigma} \sum_{\sigma'} \frac{1}{1 - (NT_{\sigma})(NT_{\sigma'}^*)} (M\ X)_{\sigma\sigma'} | \sigma \rangle \langle \sigma' |$$

$$= \sum_{\sigma} \sum_{\sigma'} \sum_{\sigma_2} \frac{1}{1 - (NT_{\sigma})(NT_{\sigma'}^*)} NT_{\sigma\sigma_2} T^*_{\sigma'(\sigma_2+\sigma'-\sigma)} X_{\sigma_2(\sigma_2+\sigma'-\sigma)} | \sigma \rangle \langle \sigma' | \qquad \text{A.18}$$

We are intersted in evaluating A.18 when the quantity X is chosen to be $\overline{| \pi_T \rangle}\ \overline{\langle \pi_T |}$. The plane wave representation of this quantity is given in III.7.10. In particular, we have

$$\overline{| \pi_T \rangle}\ \overline{\langle \pi_T |} = \sum_{\sigma_1 \sigma''} \frac{\langle \sigma_1 | p_i \rangle}{1 - NT_{\sigma_1}} \frac{\langle \sigma'' | p_i \rangle^*}{1 - NT_{\sigma''}} | \sigma_1 \rangle \langle \sigma'' |.$$

The required matrix elements are found to be

$$\langle \sigma_2 | \overline{\pi_T} \rangle\ \overline{\langle \pi_T} | \sigma_2+\sigma'-\sigma \rangle = \sum_{\sigma_1 \sigma''} \frac{\langle \sigma_1 | p_i \rangle}{1 - NT_{\sigma_1}} \frac{\langle \sigma'' | p_i \rangle^*}{1 - NT^*_{\sigma''}} \langle \sigma_2 | \sigma_1 \rangle \langle \sigma'' | \sigma_2+\sigma'-\sigma \rangle$$

$$= \frac{\langle \sigma_2 | p_i \rangle}{1 - NT_{\sigma_2}} \frac{\langle \sigma_2+\sigma'-\sigma | p_i \rangle^*}{1 - NT^*_{(\sigma_2+\sigma'-\sigma)}}. \qquad \text{A.19}$$

On substituting these values into A.18, we obtain

$$\sum_{\sigma} \sum_{\sigma'} \sum_{\sigma_2} \frac{1}{1 - (NT_{\sigma})(NT_{\sigma'}^*)} NT_{\sigma\sigma_2} T^*_{\sigma'(\sigma_2+\sigma'-\sigma)} \frac{\langle \sigma_2 | p_i \rangle}{1 - NT_{\sigma_2}} \frac{\langle \sigma_2+\sigma'-\sigma | p_i \rangle^*}{1 - NT^*_{(\sigma_2+\sigma'-\sigma)}} | \sigma \rangle \langle \sigma' |, \qquad \text{A.20}$$

Which is the result we were looking for.

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